Notes on the Ginzburg-Landau Theory

ICMR Summer School on Novel Superconductors
University of California, Santa Barbara
August 2 – August 15, 2009

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Chapter 3

Superconductivity as a macroscopic quantum phenomenon

In the modern view, superconductivity is understood as a phase ordering of pairs of electrons (Cooper paris) into a macroscopic quantum state. Let us build to this macroscopic quantum state by first considering a single pair and then turn to the many-pair state. Our approach will be heuristic. The goal is not rigor, but to convey physical insight.

3.1 The macroscopic quantum pair wave function

3.1.1 Wave function of a single pair

Consider the wave function of a pair of electrons that are bound together by some interaction.

$$\psi_p(\vec{\rho}, \vec{r}) = \Phi(\vec{\rho})\chi_{\alpha\beta}\psi(\vec{r}) \tag{3.1}$$

where

$$\Phi(\vec{\rho}) = R(\rho)Y_{lm}(\theta, \phi) \tag{3.2}$$

is the wave function of the internal (relative) coordinate of the pair and $\psi(\vec{r})$ is the wave function for the center of mass coordinate of the pair. $\chi_{\alpha\beta}$ is the spin wave function of the pair and can be either a singlet or a triplet. The exact details of the internal wave function depend on the nature of the interaction, whereas the wave function for the center of mass motion obviously does not.

Fig. 3.2 illustrates schematically the situation for s-wave ($l = 0$, spin singlet), p-wave ($l = 1$, spin triplet) and d-wave ($l = 2$, spin singlet) pairing. The indicated symmetries are consistent with the overall requirement for a change of sign under exchange of the electrons. Note that, as is customarily done in pedagogical treatments, the use of spherical harmonics here should be viewed as a surrogate for the appropriate representations of the actual symmetry of the crystal (e.g., cubic, tetragonal, etc). Note also that, in the presence of a crystal potential, the angles θ and ϕ are referenced to the crystal axes and are not free to rotate in space as is the case in superfluid He-3.

Since we are interested only in an emergent, macroscopic phenomenological theory, we need not preserve the details of the internal wave function. This suggests taking an average over the internal coordinates

$$\int \Phi(\vec{\rho})\rho^2 d\rho \, d\cos(\theta) \, d\phi = \alpha \tag{3.3}$$

which leads to the result that $\psi_p \rightarrow \alpha \, \chi_{\alpha\beta} \, \psi(\vec{r})$. It is also important to establish the size ξ_0 of the pair, because our averaged wave function only makes sense for spatial dimensions greater than ξ_0. (See Section 3.1.2 below.)

This is fine for s-wave pairing ($l = 0$), but clearly goes to far for higher angular momentum pairing, since in that case α as defined above would be identically zero. This suggests that the angular momentum wave function needs to be retained and that we should take

$$\int \Phi(\vec{\rho})\rho^2 d\rho = \alpha' \tag{3.4}$$

in which case, ψ_p becomes

$$\psi_p \rightarrow \alpha' \, Y_{lm}(\theta,\phi) \, \chi_{\alpha\beta} \, \psi(\vec{r}) \tag{3.5}$$

which is the wave function for a bosonic particle (our pair) with certain symmetry and spin properties.

3.1.2 Size of a Cooper pair

The size of a Cooper pair can be estimated using elementary considerations. The slope of the dispersion curve in the normal state at the Fermi energy is

$$\frac{\partial E(k)}{\partial k} \, |_{E_F} = \hbar v_F \tag{3.6}$$

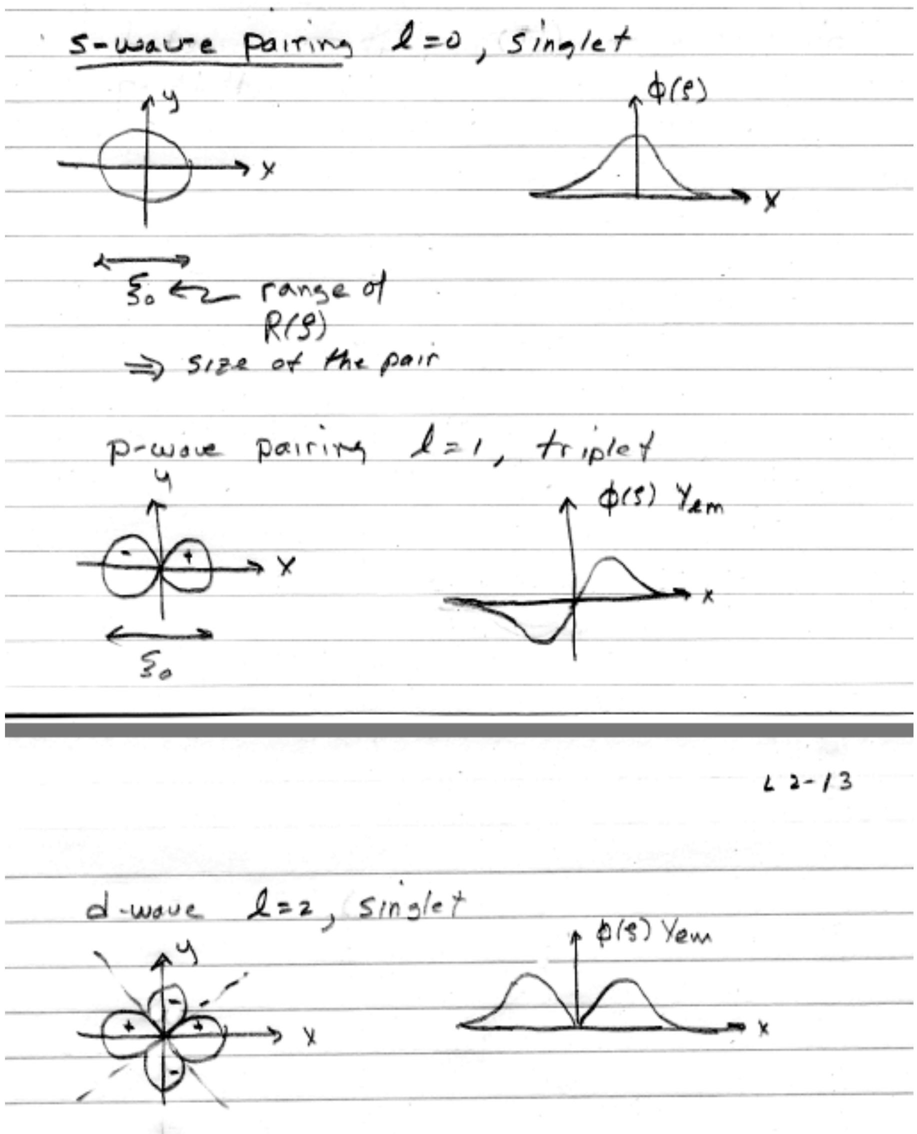

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Figure 3.1: This is the caption.

This relation permits us to relate the energy scale of states involved in pairing (the gap Δ) to the range of k-vectors involved.

$$\delta k \approx \frac{\Delta}{\hbar v_F} \tag{3.7}$$

which defines the characteristic length scale (i.e., size) of a Cooper pair

$$\xi_0 \approx \frac{1}{\delta k} \approx \frac{\hbar v_F}{\Delta} \tag{3.8}$$

This result can equally well be obtained by associating $\hbar/\Delta$ with the time scale to form a Cooper pair and asking how far the pairing electrons go in that time. The result is $\hbar v_F/\Delta$, because the motion is ballistic. The value of this alternative formulation is that it can be used to estimate the size of a Cooper pair when the superconductor is dirty (i.e., when the mean free path $l << \xi_0$).

In this case, the pairing electrons random walk with step length l for a time $\hbar/\Delta$, which using the notions of a random walk, leads to the result

$$\xi_0^d \approx \sqrt{\frac{\xi_0}{l}}\ l \approx \sqrt{\xi_0 l} << \xi_0 \tag{3.9}$$

where ξ_0/l is the number of steps taken of length l. This result leads to the very important insight that for the same interaction strength, the size of a Cooper pair is smaller in a dirty superconductor than in a clean one.

Lying behind this result, however, is the profound fact that the pairing interaction is unaffected by elastic scattering. Put more dramatically, random walking electrons pair just a effectively (i.e. with same energy Δ) as ballistic electrons. This fact is known as Anderson's Theorem and is a consequence of the fact that Cooper pairing occurs between time-reversed pairs, of which the simplest case is those with equal an opposite k-vectors. But a time-reversed random walk transverses the same path as the original path (backwards of course), and therefore they are legitimate pairing states in the BCS theory.

3.1.3 Wave function for many pairs

We have examined the internal structure of a single, bound pair of electrons. It can be understood by elementary quantum mechanics. Now we must consider a large number of pairs. The situation is very complicated indeed, because in a superconductor the size of the pair can be larger (and for the conventional, low T_c superconductors, much, much larger) than the mean separation between the pairs. Thus the pairs strongly overlap. Still in a superconductor, even under this condition, there is a strong correlation into pairs, in which the correlation is identical for each pair, and the identical wave functions for the pairs are phase coherent.

In quantum mechanics, this leads to a *coherent state*, which is essentially the classical limit of a single bosonic mode in a quantum system. The coherent state contains a macroscopic occupation (number of particles) in a single mode. In our case, it leads to a macroscopic quantum state in which all the pairs are in the same quantum state with a well-defined quantum phase. (For a helpful discussion of coherent states, see *Applied Quantum Mechanics*, Chapter 13, by W.A. Harrison).

To gain better insight into the nature of a coherent state, consider the symmetric combination of a set of N pairs, each described by a wave function ψ_i

$$\psi_N\{r_i\} = \sum' \Big\{\psi_1(r_1)\psi_2(r_2)\psi_3(r_3)......\psi_N(r_N)\Big\} \tag{3.10}$$

where the prime on the sum indicates a sum over all permuations of the product of ψ_i's (i.e., a fully symmetric wave function).

Note that

$$\psi_N \propto e^{i(\phi_1+\phi_2+....\phi_N)} \tag{3.11}$$

If the $\phi_i's$ are random, ψ_N is incoherent. If on the other hand,

$$\phi_1 = \phi_2 = \phi_3.... = \phi_N = \phi \tag{3.12}$$

we have

$$\psi_N \propto e^{iN\phi} \tag{3.13}$$

which is a coherent state.

Recall, however, that N and ϕ are quantum mechanically conjugate variables governed by an uncertainty relation

$$\Delta N \Delta\phi \geq 1 \tag{3.14}$$

Therefore, if N is fixed, ϕ is completely undetermined, which obviously is not good for a superconductor.

To get a state where ϕ is well determined and N is well enough determined, we must take a linear combination of states with different N.

$$\psi_{sc} = \sum_N \lambda_N \psi_N \tag{3.15}$$

where the weight λ_N is peaked up around some average value of N. Achieving this is not a problem, if N is large. For example, if we want $\Delta\phi \approx 10^{-10}$, it requires $\Delta N \approx 10^{10}$, but even then $\Delta N/N \approx 10^{-13}$ for a typical metal.

To make ϕ exactly determined, one must take all N from one to infinity with $\lambda_N = 1$, yielding of lambdasubN.pdf

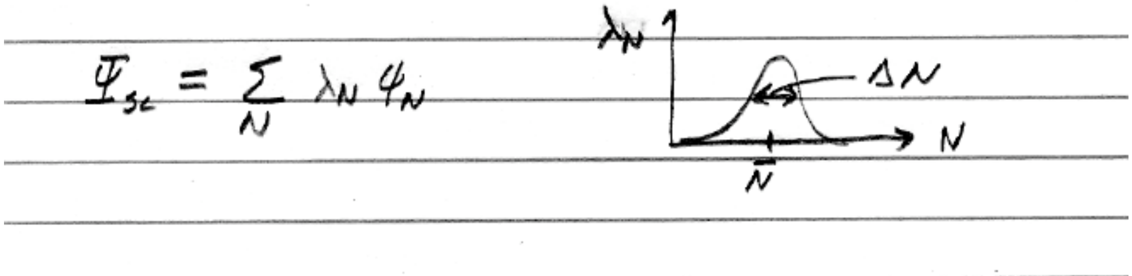

Figure 3.2: This is the caption.

$$\psi_{sc} \sim \sum_N e^{iN\phi} = \delta(\phi) \tag{3.16}$$

Of course these arguments are only suggestive, but they do capture much of what is essential. A proper theoretical treatment goes deep in to microscopic theory. It also must deal with the fact mentioned above that at least for conventional superconductors the pairs greatly overlap. At face value, the wave function in Eqn. xxx makes sense only when the pairs are relatively small and overlapping minimally. Still the basic idea is right. What happens in BCS theory is that the pairing is viewed in k-space rather than real space. The pairing is between states with equal and opposite k values (more generally and precisely, time-reversed pairs), and it is the phases of these states that order. And one takes a linear combination of states with different N for exactly the same reason as discussed above. When the result is Fourier transformed back into real space one find pairs that individually are of size ξ_0 and greatly overlapping.

So in the end, we are led to a wave function of the form

$$\psi_{sc}(\vec{r}) = Y_{lm}(\theta,\phi)\, \chi_{\alpha\beta}\, \psi(\vec{r}) \tag{3.17}$$

for the center of mass motion of the pairs.

Before ending this discussion, it will be helpful going forward to mention some issues of nomenclature. In the usual Ginzburg-Landau theory, it is $\psi(\vec{r})$ that is taken as the wave function (or order parameter, as it is sometimes called in that theory), because, as we shall see, the internal angular part does not matter for symmetry reasons in the case of an infinite sample. On the other hand, when interfaces are involved, as in the Josephson effect or when tracking the evolution of the phase going from one face of a material to another, it does matter, and technically one should use $\psi_{sc}(\vec{r})$. There could be phase changes of π after all. As is usually the case in such matters of potentially ambiguous nomenclature, the meaning is clear from the context.

3.2 Properties of the superconducting macroscopic quantum state

We now have a pair wave function $\psi(r)$ that describes the center of mass behavior of the pairs. It describes carriers of mass $m^* = 2m$ and charge $e^* = 2e$. We now turn to the physical interpretation of that wave function. For pedagogical simplicity, we ignore spin and take s-wave pairing. Subsuming the factor α into ψ, the total wave function ψ_{sc} reduces to just the center of mass part, $\psi_{sc} = \psi$. We will return to finite angular momentum pairing later.

3.2.1 The number and current density of the pairs.

The density of pairs is

$$n_s{}^* =< \psi^* \psi >= \psi^* \psi \tag{3.18}$$

and the current density

$$J_s = e^* < \psi^* J_{op} \psi >= e^* \psi^* J_{op} \psi \tag{3.19}$$

$$= \frac{e^* \hbar}{m^*} (\psi^* \nabla \psi - \psi \nabla \psi^*) \tag{3.20}$$

$$= \frac{e^* \hbar}{m^*} Im(\psi^* \nabla \psi) \tag{3.21}$$

Note that in the above the expectation values have been replaced by the quantum operator operating directly on the wave function. This is because in the macroscopic quantum limit these expectation values are c-numbers, and therefore quantum fluctuations do not matter. Quantum mechanics has been reduced to simply using the appropriate operator on a macroscopic field. The situation is not unlike going from quantum electrodynamics to Maxwell's equation, except that, as we have emphasized, $\hbar$ remains in the macroscopic equations.

Moreover, writing

$$\psi(r) = \mid \psi(r) \mid e^{i\phi(r)} \tag{3.22}$$

J_s can be written

$$\vec{J}_s = e^* \mid \psi \mid^2 \frac{\hbar \nabla \phi}{m^*} \tag{3.23}$$

permitting the identification

$$\vec{v}_s = \frac{\hbar \nabla \phi}{m^*} \tag{3.24}$$

as the velocity field of the pairs.

3.2.2 Time dependence of the wave function

Next let us consider the time dependence of ψ. This is a more subtle issue. We begin by noting that in quantum mechanics, for a stationary state,

$$\psi(t) \sim e^{-iEt/\hbar} \tag{3.25}$$

where here E is the energy. The subtle point is that in the macroscopic quantum model, it is the electro-chemical potential of the pairs

$$\mu_p = \mu_c + e^* U \tag{3.26}$$

that plays the role of the energy. Here μ_c is the chemical potential and U the electrostatic potential. While it is perhaps intuitive that μ_p is the energy of each pair, the use of a thermodynamic quantity in the wave function is not so obvious, but none the less correct in the macroscopic limit, as proved by Gorkov.

Thus, provided $\mid \psi \mid$ is constant in time, the time dependence of ψ reduces to

$$\hbar \frac{\partial \phi}{\partial t} = -E = -\mu_p \tag{3.27}$$

Note that the restriction that $\mid \psi \mid$ is constant in time is very important. When $\mid \psi \mid$ is a function of time, it means that there is a conversion between pairs and the normal excitations in the superconductor. This is a non-equilibrium process, and such process in superconductors are complicated and not describable by our macroscopic quantum model, except in some very special cases.

In any event, under the restriction that $\mid \psi \mid$ is time independent, we have

$$\frac{\partial J_s}{\partial t} = \frac{\partial}{\partial t}(e^* | \psi |^2 \frac{\hbar \nabla \phi}{m^*}) = \frac{e^* | \psi |^2}{m^*} \nabla (\frac{\hbar \partial \phi}{\partial t}) \tag{3.28}$$

$$= e^* | \psi |^2 (\frac{-\nabla \mu_p}{m^*}) \tag{3.29}$$

which implies

$$\dot{\vec{v}}_s = \frac{-\nabla \mu_p}{m^*} \tag{3.30}$$

and we see that the pairs respond inertially to the force $F = -\nabla \mu_p$.

3.2.3 Inclusion of the magnetic field

To include the magnetic field in quantum mechanics, the prescription is

$$p \rightarrow -i\hbar \nabla \tag{3.31}$$

where $p = mv + (e/c)A$ is the canonical momentum and A is the vector potential. Therefore the mechanical momentum

$$mv = (p - \frac{e}{c} A) \rightarrow -i\hbar (\nabla - \frac{ie}{\hbar c} A) \tag{3.32}$$

Formally, the rule is

$$\nabla \rightarrow (\nabla \mp \frac{ie}{\hbar c}) \tag{3.33}$$

where the minus sign is used for ψ and the plus for ψ^*.

Making this substitution for ∇ in the expressions above for J_s, and recalling that for a superconductor e becomes e^*, we are led to

$$J_s = \frac{e^* \hbar}{m^*} (\psi^* \nabla \psi - \psi^* \nabla \psi^*) - \frac{e^{*2}}{m^* c} | \psi |^2 A \tag{3.34}$$

or equivalently

$$= \frac{e^* | \psi |^2}{m^*} (\hbar \nabla \phi - \frac{e^*}{c} A) \tag{3.35}$$

and as before

$$\hbar \frac{\partial \phi}{\partial t} = -\mu_p \tag{3.36}$$

and now

$$v_s = \frac{1}{m^*}(\hbar\nabla\phi - \frac{e^*}{c}A) \tag{3.37}$$

Thus in the end, we are led to familiar equations from quantum mechanics, with the exception that the energy is replaced by the electro-chemical potential, and the operators operate directly on the wave function, not in expectation values. Put another way, $\mid \psi \mid^2 = n_s^*$ is the physical density of pairs not just a probability density, and similarly J_s is the physical current density.

As we are about to see, we will be rewarded for our formal efforts thus far.

3.3 Derivation of the hallmarks of superconductivity

The three hallmarks of superconductivity follow naturally and simply from the results above.

Zero resistance

As we found above, in the presence of forces, the acceleration of the pairs is

$$\dot{\vec{v}}_s = \frac{-\nabla\mu_p}{m^*} \tag{3.38}$$

which, as we noted, is a purely inertial response. It is the equation for a frictionless fluid (the superfluid) and clearly implies zero resistance. One might ask, but aren't there scattering processes for the pairs that would produce resistance? The answer is no. Any given pair (and concommittantly the two electrons that comprise the pair) is (are) locked with all other pairs in a macroscopic quantum state and cannot be scattered individually. While equation xx is certainly valid, it is incorrect to assume that it reflects an independent particle (here pair) response, just because it does so in the normal state. As we shall see, under some circumstances superconductors do exhibit resistance, but the processes that govern that resistance involve the full coherent state of the pairs.

The Meissner effect

The Meissner effect also follows easily. First note that

$$\nabla \times J_s = \frac{e^*|\,\psi\,|^2}{m^*}\nabla \times (\hbar\nabla\phi - \frac{e^*}{c}A) = -\frac{e^*|\,\psi\,|^2}{m^* c}B \tag{3.39}$$

which, using Maxwell's equation relating B and J

$$\nabla \times B = \frac{4\pi}{c}J \tag{3.40}$$

leads for a superconductor to

$$\nabla \times \nabla \times B = -\frac{1}{\lambda^2}B \tag{3.41}$$

or equivalently

$$\nabla^2 B = \frac{1}{\lambda^2}B \tag{3.42}$$

where

$$\lambda^2 = \frac{m^* c^2}{4\pi e^* {n_s}^*} \tag{3.43}$$

is the superconducting magnetic field penetration depth, which we note depends only on the density of pairs ${n_s}^* = |\,\psi\,|^2$.

Equation xxx clearly leads to decaying fields at the surface of a superconductor $B = H_a e^{-x/\lambda}$, which is the Meissner effect, and, in addition, we now have an expression for λ.

currents in MQM.pdf

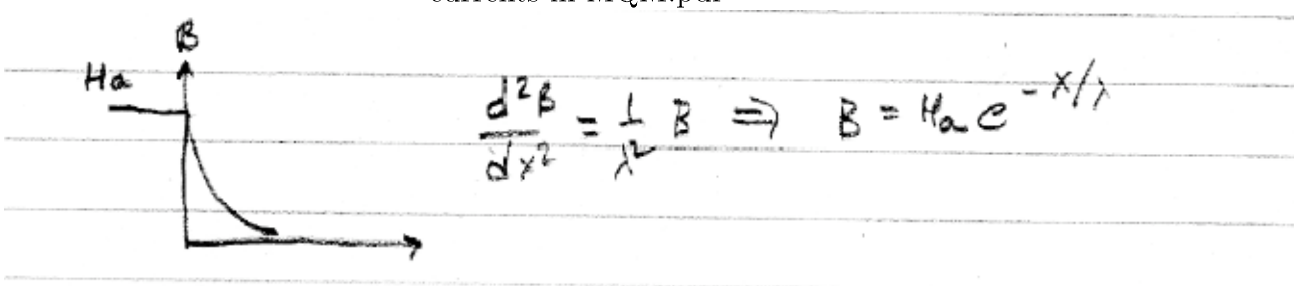

Figure 3.3:

Flux quantization

Consider a multiply connected hollow superconducting cylindar containing some trapped flux. Assume for the moment that the walls of the cylindar are thick compared to λ ($t >> \lambda$)

cylindar flux quant.pdf

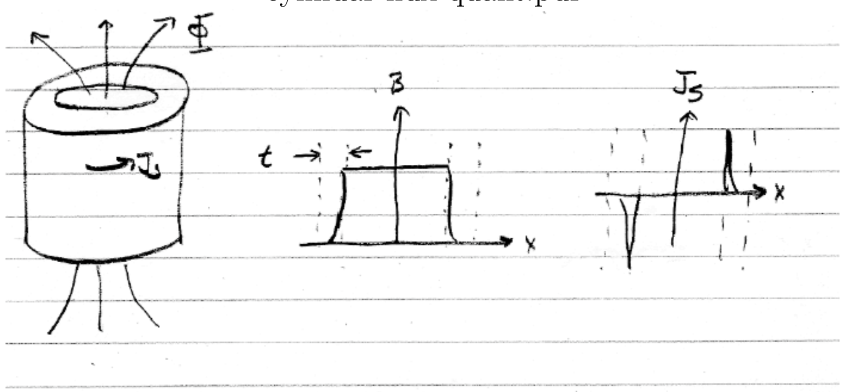

Figure 3.4:

Since the pair wavefunction $\psi(r)$ must be single-valued around any closed path, we require

$$\oint_C \nabla\phi \cdot \vec{dl} = n2\pi \tag{3.44}$$

which is just the Bohr-Sommerfeld quantization condiation $\oint p \cdot dl = nh$ applied to a superconductor.

If the thickness t of the walls of the cylindar is much smaller than λ ($t << \lambda$), a contour C can be found for which $J_s = 0$. Hence,

$$\vec{J}_s = \frac{e^* |\ \psi\ |^2}{m^*}(\hbar\nabla\phi - \frac{e^*}{c}A) = 0 \tag{3.45}$$

which yields

$$\nabla\phi = \frac{e^*}{\hbar c}\vec{A} \tag{3.46}$$

and therefore

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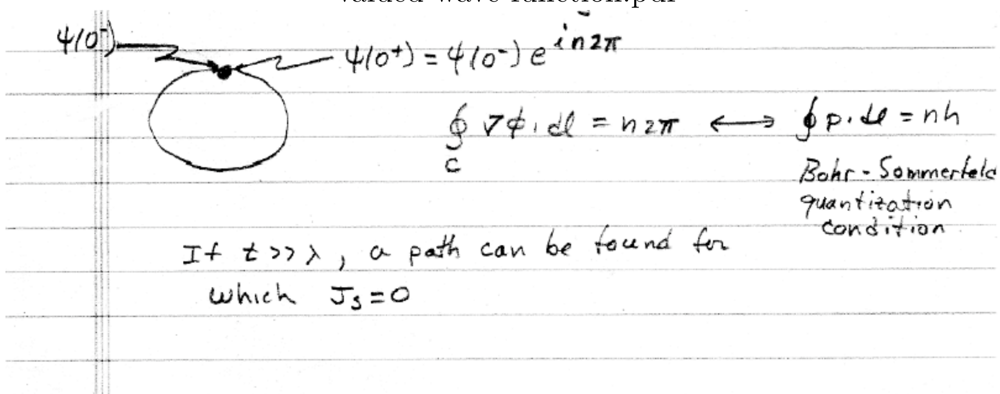


Figure 3.5:

$$\oint_C \nabla\phi \cdot \vec{dl} = \frac{e^*}{\hbar c} \oint \vec{A} \cdot \vec{dl} = n2\pi \tag{3.47}$$

and since $\oint \vec{A} \cdot \vec{dl} = \Phi$, we are led immediately to flux quantization

$$\Phi = n\frac{hc}{e^*} = \Phi_0 \tag{3.48}$$

where $\Phi_0 = hc/e^*$ is the superconducting flux quantum.

When the path C is such that $J_s \neq 0$, the above argument yields a more general quantization condition

$$\Phi' = \Phi + \frac{m^*c}{e^{*2}} \oint \frac{\vec{J_s} \cdot \vec{dl}}{\mid \psi \mid^2} = n\Phi_0 \tag{3.49}$$

which shows that in general it is the so-called fluxoid Φ' that is quantized.

Having demonstrated that the macroscopic quantum model of superconductivity accounts for the hallmarks of superconductivity – no small achievement – let us turn to a more systematic development of the model.

Chapter 4

Elaborations of the macroscopic quantum model

The macrosocpic quantum model has given us a lot. But, as it stands, it is not sufficient. There is no equation with which to calculate ψ. Specifically, without some extension, it cannot deal with situation where $\mid \psi \mid$ varies in space or even uniformly in magnitude. A powerful phenomenological way to do this is to construct the free energy of a superconductor and then apply the calculus of variation to provide a set of differential equations that describe the equilibrium behavior of ψ. This is the point of view taken in the Ginzburg-Landau theory. Also, when both ${n_s}^*$ is constant and it is not necessary to keep track of the phase of ψ, the macroscopic quantum model reduces to the London theory, which is in essence the classical limit of the macroscopic quantum model. Mainly, it is applicable to the Meissner state, but a clever extension makes it applicable to in the vortex state as well. We will begin our discussion with the traditional London theory.

Before embarking on this program, however, we must be more precise in our notation. In dealing with the magnetic properties of superconductors, it is often desirable to make a distinction between the microscopic fields and their macroscopic averages. Following convention in superconductivity, we will take h to be the microscopic field and B to be the macroscopic average of h. Consistent with this, we treat J_s as a real current (i.e., $\nabla \times h = (4\pi/c)J_s$). Although it is less important, when the distinction is critical, we will use e for the microscopic electric field and E for its macroscopic average, otherwise we will stick with E. Confusion with Planck's constant and the electric charge is possible, but the distinction is usually obvious from the context.

Also, following the convention in thermodynamics, we use the Gibbs free energy G when the magnetic field is the thermodynamic variable and the Helmholtz free energy F when it is the magnetic induction. But note that the thermodynamics of a superconductor in more interesting than just the distinction between G and F. We must include the kinetic energy of the pairs as a reversible form of energy, which has some very interesting conseqences. We address these issues at the end of this chapter.

4.1 The London theory

As noted above, when both the pair denisity $n_s{}^* = |\psi|^2$ is constant and we need not keep track of the phase of ψ, the macroscopic quantum model of superconducdtivity reduces to the London theory, which was one of the early phenomenological theories of superconductivity that remains useful today. As we have already shown, under the assumed conditions,

$$\nabla \times J_s = \frac{e^* |\psi|^2}{m^*} \nabla \times (\hbar \nabla \phi - \frac{e^*}{c} A) = -\frac{e^* |\psi|^2}{m^* c} B \tag{4.1}$$

to which we add

$$\frac{\partial J_s}{\partial t} = \frac{e^* |\psi|^2}{m^*} \hbar (\frac{\partial \nabla \phi}{\partial t} - \frac{e^*}{c} \frac{\partial A}{\partial t}) \tag{4.2}$$

$$= \frac{e^* |\psi|^2}{m^*} (-\nabla \mu_p - \frac{e^*}{c} \frac{\partial A}{\partial t}) \tag{4.3}$$

$$= \frac{e^* |\psi|^2}{m^*} E = \frac{c^2}{4\pi} \frac{1}{\lambda^2} E \tag{4.4}$$

where here

$$E = -\nabla U - \frac{1}{c} \frac{\partial A}{\partial t} \tag{4.5}$$

is the electric field, and for convenience of notation, we neglect possible gradients in μ_c.

Summarizing in concise form, we have the London equations

$$\frac{4\pi\lambda^2}{c} \nabla \times J_s = -B \tag{4.6}$$

and

$$\frac{4\pi\lambda^2}{c^2} \frac{\partial J_s}{\partial t} = E \tag{4.7}$$

Expressing J_s in terms of v_s, the London equations can also be written in the alternative, remarkably concise form

$$\nabla \times v_s = -\frac{e^*}{m^* c} B \tag{4.8}$$

and

$$\frac{\partial v_s}{\partial t} = \frac{e*}{m*} E \tag{4.9}$$

Note that $\hbar$ does not appear in these equations. We have suppressed the macroscopic quantum origins of superconductivity. We have, in effect, the "classical" limit of superconductivity.

We have already discussed the physical content of these equations (zero resistance and the Meissner effect) but a few additional points are in order. First, in their alternative form, we see that when $B = 0$ (i.e., in the Meissner state) the equations reduce to those of a frictionless, irrotational (i.e., vortex free) electronic fluid. If B were not zero, these equations imply there would have to be vorticity, which indeed is the case in a type 2 superconductor in the vortex state.

These equations also make explict the inductive nature of the intertial supercurrent response. From Eqn xxxx, we see that we can define a superconding kinetic inductivity

$$\mathcal{L}_{\mathcal{K}} = \frac{4\pi\lambda^2}{c^2} \tag{4.10}$$

Finally, although we have seen that the London theory is a natural consequence of the macroscopic quantum model, this does not do justice to London's achievement. He developed his theory as a phenomenology of superconductivity to describe zero resistance and the Meissner effect by pure physical insight, well before the underlying notion that superconductivity is a macroscopic quantum state of paired electrons was clear. But London apparently had somehing like this picture in the back of his mind, because in a footnote in his famous book *Superconductivity*, he conjucture that the flux in a superconductor might be quantized, with the charge e rather than $2e$, however.

The astute reader would ask, if London did not know about pairs, how can $e^* = 2e$ and $m^* = 2m$ appear in the London equations. An even more astute reader would note (as is self-evident from the equations in their alternative form) that the factors of two in e^* and m^* cancel out. A similar cancellation occurs in the expression for λ, if one notes that ${n_s}^*$ is half the electron denisty that has gone supercondcting. Consequently, the equations could just as well be written in terms of e, m and the electron density n, which is what London did. We use e^*, m^* and ${n_s}^*$ because we arrived at these equations from the macroscopic quantum point of view. Of course, in the end, one can simply take λ from experiment.

4.2 The Ginzburg-Landau theory

It is physically reasonable to assert that the macroscopic quantum wave function of a superconductor is that function which minimizes the free energy. Following Ginzburg and Landau, we can consider F_s to be a functional of $\psi(r)$ and use the calculus of variations to derive a set of differential equations for $\psi(r)$ that governs its behavior under general equilbrium conditions. Toward this end, it is first necessary to construct an expression for the free energy.

4.2.1 Construction of the free energy

From our considerations thus far, it is clear that the Helmholtz free energy of a superconductor has three main contributions.

$$F_s = F_n + \text{Condensation energy} + \text{Kinetic energy} + \text{Field energy} \tag{4.11}$$

which illustrates the general principle that the behavior of a supercondcutor is a trade off between the kinetic and field energies and the condensation energy.

Using the results from the macroscopic quantum model, we can get an expression for the free energy

$$F_s = F_n + \int d^3r \Big\{ f_0 + \frac{1}{2m^*} \left| (-i\hbar\nabla - \frac{e^*}{c}A)\psi \right|^2 + \frac{h^2}{8\pi} \Big\} \tag{4.12}$$

where $f_0 = -{H_c}^2/8\pi$ is the condensation free energy denisty.

To get a better feel for the kinetic energy density term, take $\psi(r) = \mid \psi(r) \mid e^{i\phi(r)}$. Straight forward manipulation yields

$$KE = \frac{1}{2m^*} \Big\{ \hbar(\nabla \mid \psi \mid)^2 + \mid \psi \mid^2 (\hbar\nabla\phi - \frac{e^*}{c}A)^2 \Big\} \tag{4.13}$$

which shows explictly that the kinetic energy density includes contributions from both gradients in $\mid \psi \mid$ and from the usual kinetic energy. The gradient term will introduce new physics not included in the London theory.

Free energy in the London limit

To make the connection to the London theory explicit, if ${n_s}^*$ is a constant

$$F_s = F_n = \int d^3r\{f_0 + {n_s}^* \frac{1}{2} m^* {v_s}^2 + \frac{1}{8\pi} h^2\} \tag{4.14}$$

which is intuitively appealing.

Also, using $v_s = J_s / {n_s}^* e^*$ and the defintion of λ, the kinetic energy term can be written

$${n_s}^* \frac{1}{2} m^* {v_s}^2 = \frac{1}{2} \frac{4\pi\lambda^2}{c^2} {J_s}^2 = \frac{\lambda^2}{8\pi} (\nabla \times h)^2 \tag{4.15}$$

and the free energy can be written in the particularly compact and useful form

$$F_s = F_n = \int d^3r \Big\{ - \frac{{H_c}^2}{8\pi} + \frac{1}{8\pi} [\lambda^2 (\nabla \times h)^2 + h^2] \Big\} \tag{4.16}$$

Finally, note that the kinetic energy density can also be written

$${n_s}^* \frac{1}{2} m^* {v_s}^2 = \frac{1}{2} \mathcal{L}_{\mathcal{K}} {J_s}^2 \tag{4.17}$$

which emphasizes that the kinetic energy is an inductive stored energy.

The Ginzburg-Landau free energy

In this framework, however, it is not sufficient to treat the condensation energy as a constant. Clearly it must depend on the density of superconducting pairs. Therefore we generalize the free energy discussed above to

$$F_{GL}\{\psi(r), \psi(r)^*, A\} = F_n + \int d^3r \Big\{ f_0(\mid \psi \mid^2) + \frac{1}{2m^*} \Big| \, (-i\hbar\nabla - \frac{e^*}{c} A)\psi \, \Big|^2 + \frac{(\nabla \times A)^2}{8\pi} \Big\} \tag{4.18}$$

where now f_0 explicitly depends on $\mid \psi \mid^2$.

The functional form of $f_0(\mid \psi \mid^2)$ is not known a priori, so Ginzburg and Landau used a Taylor series expansion in powers of $\mid \psi \mid^2$. Indeed, the kinetic energy term can be thought of as the leading gradient term in a Taylor series, which was the point of view taken by Ginzburg and Landau, in absence of explict knowledge of the macroscopic quantum concept, although clearly they had something like this in mind. As they noted, such a Taylor series expansion should be valid near T_c, where $\mid \psi \mid^2$ is small and, as we shall see, $\psi(r)$ is slowly varying in space. Clearly this point of view will break down if $\psi(r)$ varies on a length scale comparable to the size of a Cooper pair ξ_0. These caveats notwithstanding, the theory is qualitatively useful far more generally, and it is quantitatively useful over a substantial range of reduced temperature $t = T/T_c$ below T_c.

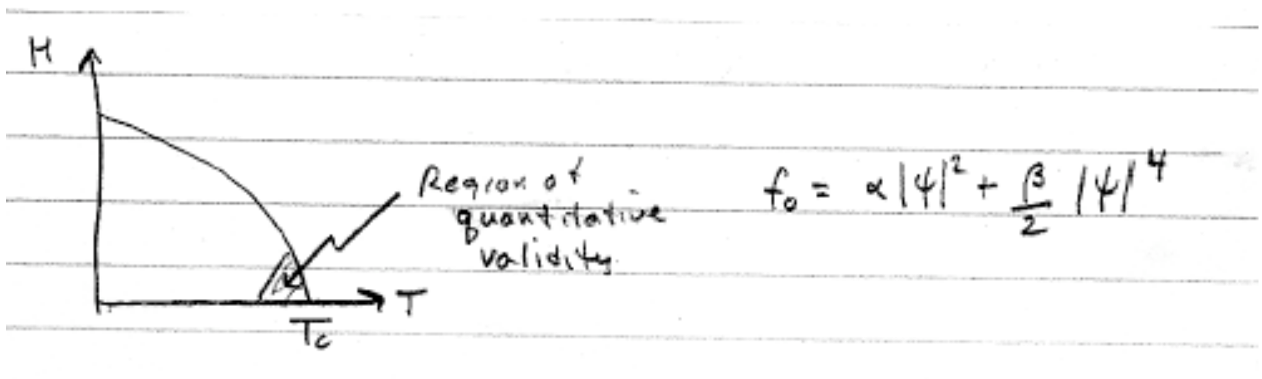


Figure 4.1:

Explicitly, Ginzburg and Landau took

$$f_0 = \alpha| \psi |^2 + \frac{\beta}{2}| \psi |^4 \tag{4.19}$$

where α and β are material dependent phenomenological parameters to be determined experimentally. Today we know that, in the case of classic BCS superconductors, α and β can also be calculated from the microscopic theory.

4.2.2 Equilbrium value of $| \psi |$ and determnation of the material prameters

As noted above, the parameters of the Ginzburg-Landau theory can be determined from experiment. Neglecting fields and currents, we know that the minimum of $f_0(| \psi |^2)$ must give the equilibrium condensation energy and the equilibrium pair density. The former can be measured directly, and the latter can be determined from a measurement of the penetration depth.

Consider the structure of f_0 as a function of ψ, depending of the signs of α and β, as illustrated in Fig. 4.2. Clearly, only $\beta > 0$ is physical. Moroever, when $\alpha > 0$, clearly the minimum (i.e., the equilbrium state) corresponds to the normal state,

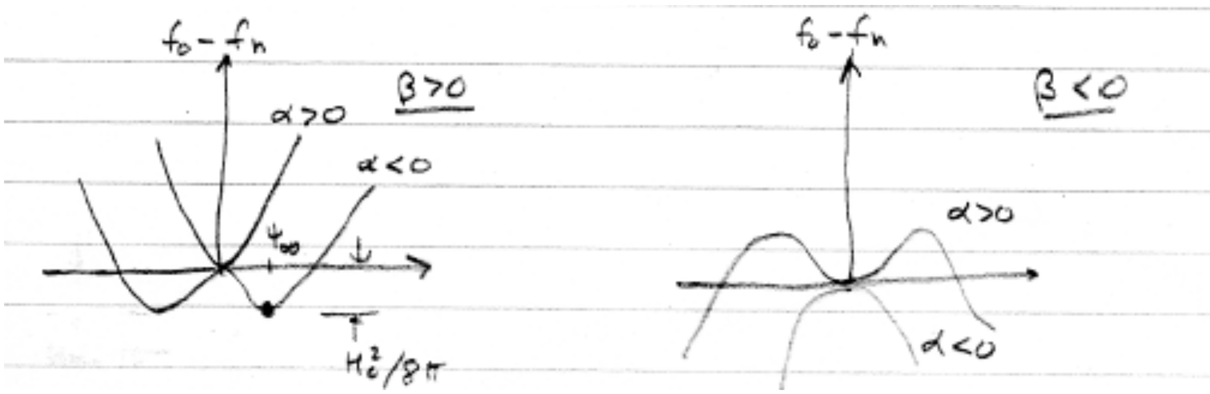


Figure 4.2: This is the caption

$|\psi_{eq}| = 0$. However, when $\alpha < 0$, the superconducting state is stable, $|\psi_{eq}| > 0$. We conclude, therefore, that $\alpha(T)$ must change sign at $T = T_c$. Following Ginzburg and Landau, we take

$$\alpha(T) = \alpha_0(\frac{T - T_c}{T_c}) = \alpha_0(t - 1) \tag{4.20}$$

and β independent of temperature. As we shall see these choices are consistent with experiment.

Note that in the figure above, when viewed as functions of ψ as opposed to $|\psi|$, the curves are sections through the full free energy density space. In this case the single x-axis becomes a plane in $Re\,\psi$ and $Im\,\psi$ space, and the full free energy curve is just a rotation of the curve shown around the z-axis. The point is that the overall phase of the wave function is arbitrary.

It is trivial to determine the equilibrium value of $|\psi|$ in the superconducting state ($\alpha < 0$), which is traditionally designated as ψ_∞ (See Fig. 4.2). Taking

$$\frac{\partial f}{\partial |\psi|} = 2\alpha |\psi| + 2\beta |\psi|^3 = 0 \tag{4.21}$$

to find the minimum, yields

$${\psi_\infty}^2 = -\frac{\alpha}{\beta} = \frac{|\alpha|}{\beta} \tag{4.22}$$

In addtion, the equilibrium condensation energy density is given by

$$(f_s - f_n) \mid_{|\psi|=\psi_\infty} = -\frac{\alpha^2}{2\beta} \tag{4.23}$$

These expressions can be easily related to experiment in order to determine α and β. Since

$$\lambda^2 = \frac{m^* c^2}{4\pi e^{*2} n_s^*} \tag{4.24}$$

it follows that

$$\psi_\infty{}^2 = \frac{\mid \alpha \mid}{\beta} = \frac{m^* c^2}{4\pi e^{*2} \lambda^2} \tag{4.25}$$

and directly, we have

$$(f_s - f_n) \mid_{|\psi|=\psi_\infty} = -\frac{\alpha^2}{2\beta} = -\frac{H_c{}^2}{8\pi} \tag{4.26}$$

Taken together, these two results yield

$$\mid \alpha \mid = \frac{e^{*2} H_c{}^2 \lambda^2}{m^* c^2} \propto (1 - t) \tag{4.27}$$

and

$$\beta = \frac{4\pi e^{*4} H_c{}^2 \lambda^4}{m^* c^4} \propto constant \tag{4.28}$$

Note that the temperature dependences of α and β determined from experiment are consistent with those assumed initially, confirming those assumptions.

4.2.3 The Ginzburg-Landau equations

Everything is now in order to derive the famous Ginzburg-Landau equations. Pulling together all the previous results, we have for the Ginzburg-Landau free energy

$$F_{GL}\{\psi(r), \psi(r)^*, A\} = F_n + \int d^3r \left\{ \alpha \mid \psi \mid^2 + \frac{\beta}{2} \mid \psi \mid^4 + \frac{1}{2m^*} \mid (-i\hbar\nabla - \frac{e^*}{c} A)\psi \mid^2 + \frac{(\nabla \times A)^2}{8\pi} \right\} \tag{4.29}$$

Note that in writing this equation, we have been a bit more precise as to what are the correct thermodynamic variables. Specifically, we include both ψ and ψ^*.

Rigorously this is necessary since ψ is a complex number and therefore has a real and an imaginary part (i.e., it really represents two functions). This distinction has no marked importance in what is to immediately follow, but if we were to use the modulus and the phase of ψ as the two functions, new insights on the thermodynamics of superconductivity are obtained. We will return to these important subtleties later.

Now taking variations of $F_s\{\psi(r), \psi(r)^*, A\}$ with respect to ψ, ψ^* and A, and after some standard manipulations, we arrive at the Ginzburg-Landau (or GL for short) equations

$$\alpha\psi + \beta|\, \psi\, |^2\psi + \frac{1}{2m^*}(-i\hbar\nabla - \frac{e^*}{c}A)^2\psi = 0 \tag{4.30}$$

and

$$J_s = \frac{c}{4\pi}\nabla \times h = \frac{e^*\hbar}{2m^*i}(\psi^*\nabla\psi - \psi^*\nabla\psi^*) - \frac{e^{*2}}{m^*c}|\, \psi\, |^2 A \tag{4.31}$$

or

$$J_s = \frac{e^*|\, \psi\, |^2}{m^*}(\hbar\nabla\phi - \frac{e^*}{c}A) = e * |\, \psi\, |^2 v_s \tag{4.32}$$

Note that the equation for ψ^* is identical to that for ψ with complex conjugation, so it contains no new information and therefore is not explicitly shown.

From the variational process, we also get an expression for the boundary conditions at any superconductor/vacuum interface

$$(-i\hbar\nabla - \frac{e^*}{c}A)\psi\, |_{\hat{n}} = 0 \tag{4.33}$$

which ensures that $\vec{J}_s \cdot \hat{n} = 0$ at the surface.

Physical content of the Ginzburg-Landau equations

Pondering these equations, we see that the second GL equation (for J_s) is exactly that obtained in the macroscopic quantum model, as it must be, only here we see that it is consistent with minimization of the free energy. It is the first GL equation that contains the new physics. Specifically, it governs the spatial variations of ψ. Associated with these spatial variations is a new length scale – the so-called GL coherence length $\xi(T)$ – that governs the distance over which ψ naturally varies (See Chapter xxx).

At the same time, however, note that the GL equations are coupled differential equations and must be solved self consistently. For example, when $\psi \neq \psi_\infty$, the second GL equation will adjust the superconducting magnetic penetration depth

$$\lambda^2 = \frac{m^* c^2}{4\pi e^* |\ \psi\ |^2} \tag{4.34}$$

self consistently. This will happen, for example, if the current density gets very high (Again, see Chapter xxx).

Note also that while the boundary condition is physically reasonable in that it ensures that no current flows through a vacuum interface, this is not the boundary condition of microscopic quantum mechanics, which requires that $\psi = 0$ at the interface. There is no problem here, however, since our ψ is a macroscopic quantum wave function in which we have averaged over the internal wave function of the pairs. The truly microscopic wave function (as, say, in the BCS theory) does go to zero at a vacuum surface. But all this happens on a length scale smaller than the size of a Cooper pair and therefore is beyond the content of the GL theory.

4.3 GL theory for non-s-wave pairing

Let us consider what happens to the GL theory when we have non-s-wave pairing. Recall that in general for our macroscopic quantum wave function, we had

$$\psi_{sc}(\vec{r}) = Y_{lm}(\theta,\phi)\, \chi_{\alpha\beta}\, \psi(\vec{r}) \tag{4.35}$$

Therefore, in the general case, it is this wave function that should be used to construct the GL free energy. Specifically, we have for the GL free energy density

$$\mathcal{F}_{GL}\{\psi_{sc}(r);\theta,\phi\} = \mathcal{F}_n + \Big\{\alpha'\, Y^*_{l,m} Y_{l,m} |\ \psi\ |^2 + \frac{\beta'}{2} Y^*_{l,m} Y_{l,m} Y^*_{l,m} Y_{l,m} |\ \psi\ |^4 \tag{4.36}$$

$$+\frac{1}{2m^*} Y^*_{l,m} Y_{l,m} |\ (-i\hbar\nabla - \frac{e^*}{c} A)\psi\ |^2 + \frac{(\nabla \times A)^2}{8\pi}\Big\} \tag{4.37}$$

where now the un-averaged internal coordinates appear explicitly, and where for simplicity we ignore spin.

As is immediately obvious, the $Y_{l,m}$'s appear only as products that are positive definite. Therefore, we can integrate over the internal coordinates safely to obtain

$$F_{GL}\{\psi_{sc}(r)\} = F_n + \int d^3r \Big\{ \alpha |\, \psi \,|^2 + \frac{\beta}{2} |\, \psi \,|^4 + \frac{1}{2m^*} |\, (-i\hbar\nabla - \frac{e^*}{c} A)\psi \,|^2 + \frac{(\nabla \times A)^2}{8\pi} \Big\} \tag{4.38}$$

where $\alpha = \alpha'$ and $\beta = \beta' \int d(cos\theta)\, d\phi\, \{Y^*_{l,m} Y_{l,m} Y^*_{l,m} Y_{l,m}\}$. In short, we have a free energy of the original GL form. Clearly, the internal symmetry does not matter.

Fundamentally, this result reflects the fact that $\mathcal{F}_{GL}$ is an observable and therefore must have the same symmetry at the crystal. Moreover, it shows that the GL theory is the same for pairs of any internal angular momentum state (s, p, d, etc). Fig. 4.3 shows the situation pictorially. Of course, this symmetry argument breaks down at surfaces and interfaces where the crystal symmetry is in general lower. For example, contrast the two paths shown in Fig. 4.4, where we show the situation for a d-wave superconductor. Going around an internal closed path involves no phase change, and therefore the distinction between ψ and ψ_{sc} is not important. By contrast, it is very important if the path goes from one surface to another.

s-wave GL.pdf

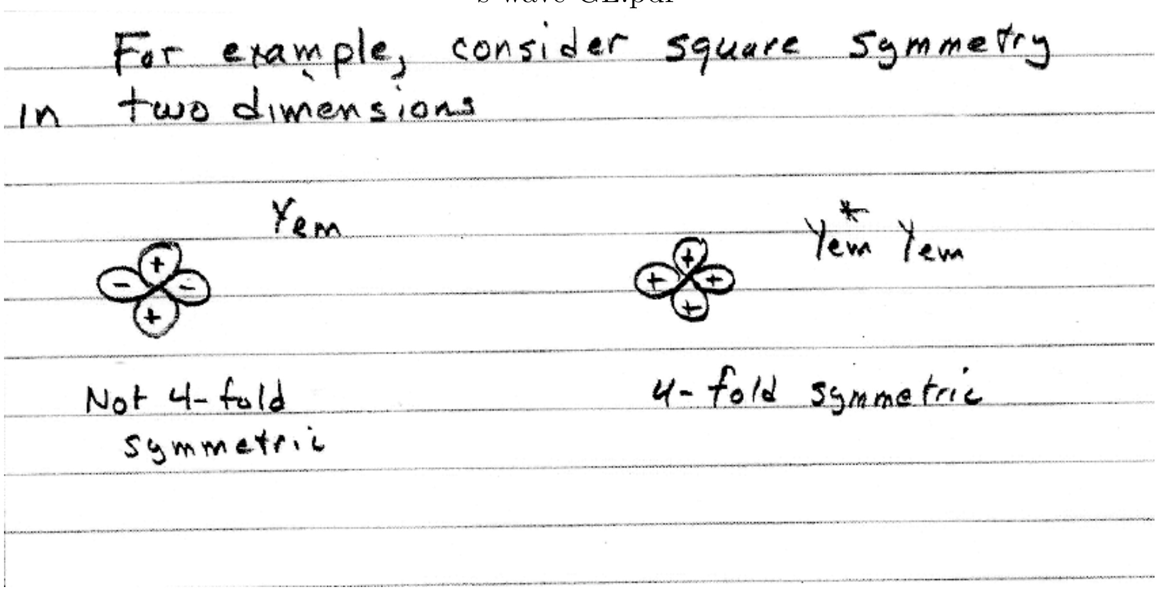


Figure 4.3: This is the caption

Actually, a similar situation arises in a more familiar context. In developing the Landau theory (See Section 4.4 immediately below) for ferromagnets and antiferromagnets, the order parameter (equivalent to our center of mass wave function) is the same for both types of order. For a ferromagnet, it is the magnetization, but for an antiferromagnet it is the amplitude of the staggered magnetization (a function that goes up,down, up, down etc as one moves through space). In our case, $Y_{l,m}$ is like the staggered magnetization.

s interior vs surface.pdf

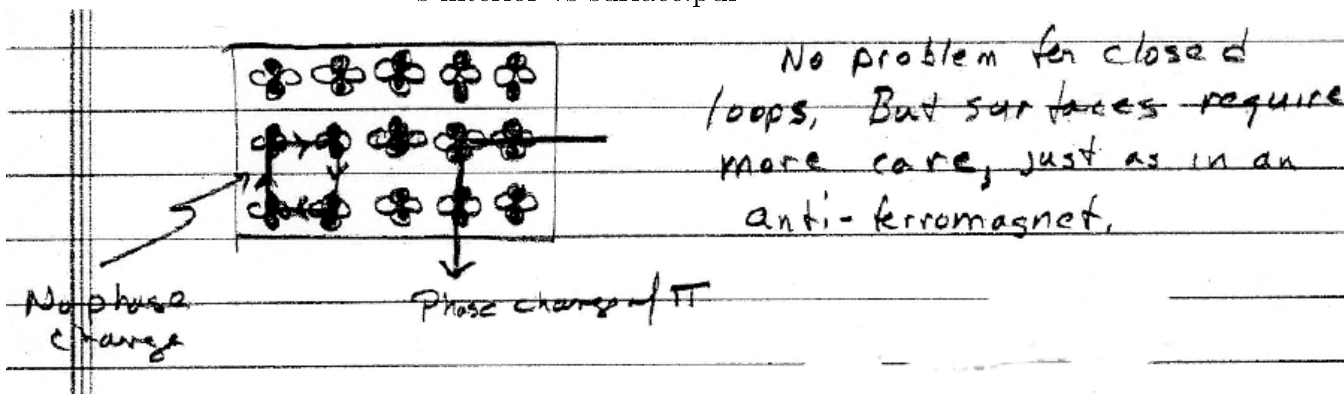


Figure 4.4: This is the caption

4.4 The Landau theory of second-order phase transitions

We have derived the GL theory from our macroscopic quantum model. Ginzburg and Landau used a different approach – one based on Landau's general theory of second-order phase transitions, of which superconductivity is only one example. The Landau theory involves the powerful concept of broken symmetry, i.e., the notion that when going through a second-order phase transition the system spontaneously generates an order of some kind that breaks a symmetry of the parent state. The classic example is the spontaneous ordering of the spins of a ferromagnet along a specific direction when the paramagnetic parent state is completely rotationally symmetric. Given a particular kind of ordering, one defines an order parameter (the magnetization in our example) and constructs the free energy of the system in terms of a Taylor series expansion (including gradient terms) of the free energy in terms of the order parameter.

In constructing the free energy, all terms consistent with the symmetries of the crystal and the nature (scalar, vector, phasor, etc) and dimensionality (one, two, three, etc) of physical space and of the order parameter must be included. All the coefficients on the expansion are to be determined experimentally. Since a second-order phase transition is continuous by its nature and the order grows continuously from zero, one can always carry out such a Taylor series expansion. Variation of this free energy in terms of the order parameter leads to a set of differential equations that describe the behavior of the system.

The Landau approach is extremely general and powerful. As it has been developed over the years, we now know that various types of order come in universality classes – the emergent behavior – that depend only on the nature of the order but not the details at the microscopic level. It does not take much imagination to see that the GL theory as developed above would lend itself to such an approach, and many treatments of superconductivity adopt this approach. In this approach, the notion of superconductivity as a phenomenon that breaks gauge invariance is front and center. We chose to emphasize the specific character of superconductivity as a macroscopic quantum phenomenon. Still, any serious student of superconductivity should be familiar with the Landau theory. Many fine treatments of this theory and its application are available.

Finally, before leaving the subject of the Landau theory, we should note that our treatment of non-s-wave pairing in Section 4.3 was very simplified and hardly exhaustive. The possible internal symmetries of the Cooper pairs are far richer than simply higher angular momentum states (e.g., breaking time reversal symmetry). Similarly, the symmetries of the crystals in which superconductivity is found can be more complex (and interesting) than in our illustrations (e.g., crystals without inversion symmetry). For these situations, the Landau theory is the best way to make progress simply.

4.5 Thermodynamic variables in superconductivity

In thermodynamics, we know that one uses the Gibbs or Helmholtz free energy depending on what is the independent thermodynamic variable (e.g., P versus V, or H versus B). P and V, and B and H are thermodynamically conjugate variables. Specifically, in the case of magnetic fields, we have for the free energy densities,

$$f = f(B) \quad \text{and} \quad g = g(H) \tag{4.39}$$

which are related by

$$g = f - \frac{HB}{4\pi} \tag{4.40}$$

The above relations are completely general and therefore apply to superconductors. What is new in the case of superconductivity is that the kinetic energy is also a reversible thermodynamic quantity. This brings some new twists.

Recall that in the GL theory we took ψ and ψ^* as the relevant variables and therefore

$$f_{GL} = f_{GL}\{\psi, \psi^*\} \tag{4.41}$$

The situation is more interesting when we write $\psi = \mid \psi \mid e^{i\phi}$. From the result

$$f_{GL} = f_n + \alpha \mid \psi \mid^2 + \frac{\beta}{2} \mid \psi \mid^4 + \frac{1}{2m^*} \mid (-i\hbar\nabla - \frac{e^*}{c}A)\psi \mid^2 + \frac{(\nabla \times A)^2}{8\pi} \tag{4.42}$$

where the kinetic energy term can be written

$$\frac{1}{2m^*} \mid (-i\hbar\nabla - \frac{e^*}{c}A)\psi \mid^2 = \frac{1}{2m^*}\{\hbar(\nabla \mid \psi \mid)^2 + \mid \psi \mid^2 (\hbar\nabla\phi - \frac{e^*}{c}A)^2\} \tag{4.43}$$

and where as usual

$$\frac{1}{m^*}(\hbar\nabla\phi - \frac{e^*}{c}A) = v_s \tag{4.44}$$

we see that the appropriate choice of thermodynamic variables are $\mid \psi \mid^2$ and $(1/m*)(\hbar\nabla\phi - e^*A) = v_s$.

That $f_s = f_s(v_s)$ is even more explicit in the London limit where

$$f_L = f_n + f_0 + {n_s}^* \frac{1}{2} m^* {v_s}^2 + \frac{B^2}{8\pi} \tag{4.45}$$

where n_s^* is constant.

The thermodynamic conjugate variable to v_s is the superconducting current density J_s

$$\frac{\partial f_s}{\partial v_s} = \frac{m^*}{e^*} J_s \tag{4.46}$$

and therefore

$$g(J_s) = f - \frac{m^*}{e^*} v_s J_s \tag{4.47}$$

Sometimes, however, it is necesary to work where the phase of the wave function is explicit. To do this, it is helpful to define a new function, the gauge-invariant phase difference γ, given by the line integral

$$\gamma \equiv \int (\nabla\phi - \frac{e^*}{\hbar c} A) \cdot \vec{dl} = \int \frac{m^* v_s}{\hbar} \cdot \vec{dl} \tag{4.48}$$

When $A = 0$,

$$\gamma = \Delta\phi \equiv \int \nabla\phi \cdot \vec{dl} \tag{4.49}$$

which motivates the nomenclature. But $\Delta\phi$ is not in general a gauge invariant quantity, where as γ is. In any event, γ is the physically relevant quantity because it is related to v_s, which is an observable. This connection is explicit in the relation

$$\hbar\nabla\gamma = m * v_s \tag{4.50}$$

which is completely general.

The point is that, although ϕ is an intuitively clear quantity, it is not the fundamental one. Indeed, when we say superconductivity is a phase ordering of Cooper pairs, technically we mean gauge invariant phase ordering, which is the basis for the erudite statement that superconductors break gauge symmetry.

The transcription to the case where $\nabla\gamma$ is the thermodynamic variable is now obvious

$$\frac{\partial f_s}{\partial(\nabla\gamma)} = \frac{\hbar}{e^*} J_s \tag{4.51}$$

and therefore

$$g(J_s) = f - \frac{\hbar}{e^*} \nabla\gamma J_s \tag{4.52}$$

While the introduction of the quantity γ may all seem a bit formal, it has real utility and importance. Consider a superconductor to which a current I is applied.

The reversible free energy delivered to the superconductor is

$$dF = IV dt \tag{4.53}$$

but

for free energy calc.pdf

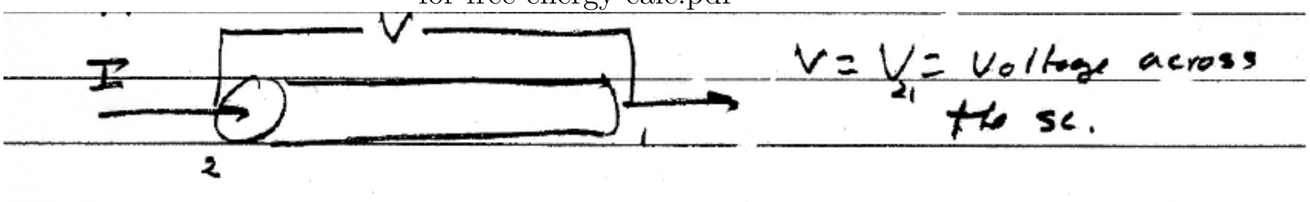


Figure 4.5: This is the caption

$$V = \int E \cdot \vec{dl} = \int (\frac{-\nabla \mu_p}{e^*} - \frac{1}{c}\frac{\partial A}{\partial t}) \cdot \vec{dl} = \int (\frac{\hbar}{e^*}\frac{\partial \nabla \phi}{\partial t} - \frac{1}{c}\frac{\partial A}{\partial t}) \cdot \vec{dl} \tag{4.54}$$

$$= \frac{\hbar}{e^*}\frac{\partial \gamma}{\partial t} \tag{4.55}$$

and therefore

$$dF = IVdt = \frac{\hbar}{e^*} I \frac{\partial \gamma}{\partial t} dt = \frac{\hbar}{e^*} I d\gamma \tag{4.56}$$

which demonstrates that γ and I are thermodynamically conjugate variables. Note that here the line integral determining γ is taken from the entire length of the superconductor (from point 1 to 2 in the figure).

Summarizing, we have for the total free energy

$$F_s = F_s(\gamma) \quad \text{and} \quad I = \frac{\hbar}{e^*}\frac{dF_s}{d\gamma} \tag{4.57}$$

and correspondingly

$$G_s(I) = F - \frac{\hbar}{e^*}\gamma I \tag{4.58}$$

Note that we are dealing here with the total free energy in terms of the extensive variables γ and I. This last relation will be of great utility in discussing the Josephson effect. When we come to the Josephson effect, we will also see that ψ and ψ^* are thermodynamically conjugate variables.

Chapter 6

Physical consequences of the Ginzburg-Landau theory

The GL theory is rich in physical consequences. We touched very briefly on some of these in the previous chapter, at least in qualitative terms. In this chapter, we discuss these consequences more comprehensively and in detail.

6.1 The GL coherence length

If $|\psi|$ deviates from its equilibrium value ψ_∞, there is a natural healing length back to the equilibrium value. This length is called the GL coherence length. Assume there are no fields or currents present, then ψ can be taken to be real. Now suppose that ψ deviates from ψ_∞

length.pdf

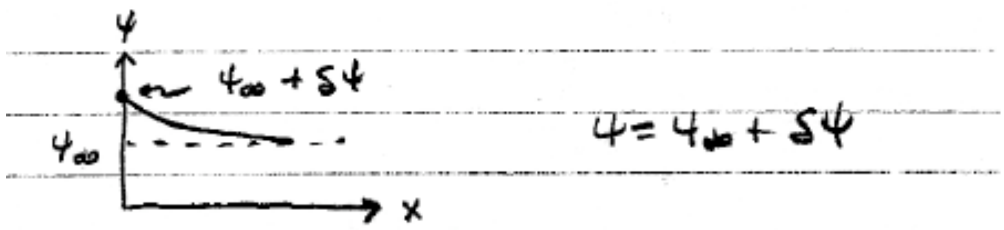


Figure 6.1: This is the caption

$$\psi = \psi_\infty + \delta\psi \tag{6.1}$$

Substituting this ψ into the first GL equation (Eqn xxx) and linearizing, we obtain,

$$2\alpha\,\delta\psi + \frac{\hbar^2}{2m*}\nabla^2\delta\psi = 0 \tag{6.2}$$

or

$$\delta\psi + \frac{\xi^2(T)}{2}\nabla^2\delta\psi = 0 \tag{6.3}$$

which has solutions of the form

$$\delta\psi \sim e^{-\frac{x}{\sqrt{2}\xi(T)}} \tag{6.4}$$

where

$$\xi(T)^2 = \frac{\hbar^2}{2m^* \mid \alpha \mid} = \xi^2(0)\Big(\frac{T_c}{T_c - T}\Big) \tag{6.5}$$

is the so-called GL coherence length. Note that $\xi(T)$ diverges as $(1-t)^{-1/2}$ as $t \to 1$.

The length $\xi(0)$ is known as the zero-temperature GL coherence length. Of course, the GL theory is not valid at $T = 0$, and the logic of the name comes from the fact that $\xi(0)$ is the zero temperature extrapolation of the GL result. The physical importance of $\xi(0)$ is that it sets the basic length scale of the healing process.

A natural question then is: what is the relation between the zero-temperature GL coherence length $\xi(0)$ and the BCS coherence length ξ_0? This question can be answered within Gorkov's formulation of the BCS theory. As shown by Gorkov, for conventional low-T_c superconductors,

$$\xi(0) = 0.74\xi_0 \qquad \text{In the clean limit, } l >> \xi_0 \tag{6.6}$$

and

$$\xi(0) = 0.85\sqrt{\xi_0 l} \qquad \text{In the dirty limit, } l << \xi_0 \tag{6.7}$$

These results are quite natural, given our discussion of the size of a Cooper pair in Section 3.1.2. Clearly the overall length scale for healing is the size of a Cooper pair. Of course the divergence of $\xi(T)$ as $T \to T_c$ means the slowly varying approximation of the GL theory is always valid near T_c.

Finally, we note that there is a very useful relation between ξ and λ. Recall that $\alpha = e^{*2}H_c^2\lambda^2/m^*c^2$, which means that

$$\xi(T) = \frac{\Phi_0}{2\sqrt{2}\pi H_c(T)\lambda(T)} \tag{6.8}$$

or

$$\xi(T)\lambda(T) = \frac{\Phi_0}{2\sqrt{2}\pi H_c(T)} \tag{6.9}$$

One useful application of this relation is to use the mean free path dependence of ξ to establish the mean free path dependence of λ. To do this first recall that $\lambda(t) \approx \lambda_0/\sqrt{1-t^4}$, which near T_c goes as $\lambda_0/2\sqrt{1-t} = \lambda(0)/\sqrt{1-t}$, where

$$\lambda(0) = \frac{\lambda_0}{2} \tag{6.10}$$

is the zero-temperature GL penetration depth, consistent with the definition of $\xi(0)$.

Now using Eqn xxx above and recalling that H_c does not depend on mean free path (due to the time-reversed nature of Cooper pairing), we have directly that

$$\frac{\lambda(0)_{dirty}}{\lambda(0)_{clean}} \approx \frac{\xi(0)_{clean}}{\xi(0)_{dirty}} \approx \sqrt{\frac{\xi_0}{l}} \tag{6.11}$$

which is consistent with the sum rule argument presented in Section 5.6. And again we emphasize that in the dirty limit, λ increases and, concomitantly, the pair density decreases.

6.2 Local versus nonlocal electrodynamics

The relative sizes of ξ and λ affect the electrodynamics of a superconductor. In the macroscopic quantum model, we average out the internal structure of a Cooper pair. But when $\lambda < \xi$, this assumption is not valild, because in this case the superconducting shielding currents are decaying over a length scale smaller than the size of a Cooper pair (See Fig. 6.2 below) . Obviously, in this situation, one must resort to microscopic theory. In the parlance of the field, we refer to these two regimes as local and nonlocal elelectrodynamics. More specifically, in the local limit, the pair current is a simple function of position (as in the macroscopic quantum model as we have developed it), whereas in the nonlocal limit, it is an integral over a region the size of a Cooper pair around any given position. Fortunately, for the vast majority of superconductors $\lambda > \xi$, and the much simpler local theory applies.

electrodynamics.pdf

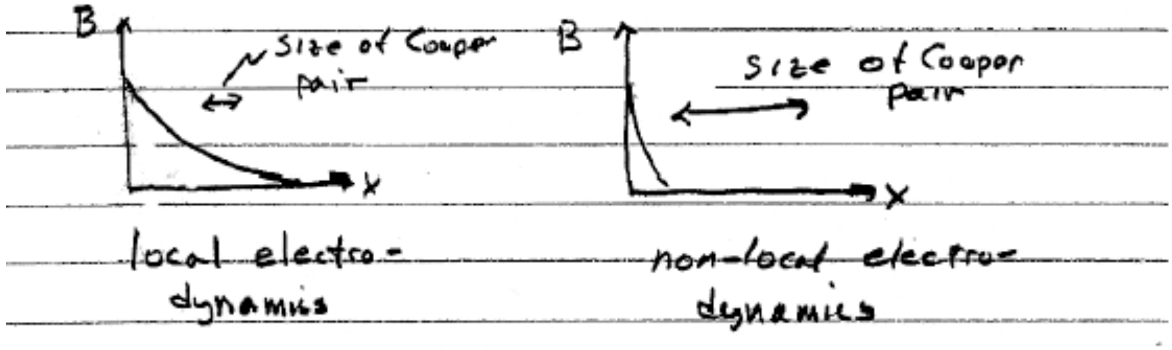


Figure 6.2: This is the caption

6.3 The proximity effect

The proximity effect arises at the interface between a superconductor and a non-superconducting material, typically a normal metal. The point is that the Cooper pairs can move back and forth across the interface. The result is that, near the interface, superconductivity is induced in the normal metal and the superconductivity in the superconductor is weakened.

Right at the interface, the situation is complicated and requires a microscopic description. At the microsopic level, the critical processes involve so-called Andreev reflections, which is a fascinating process by which normal electrons and Cooper pairs convert one into the other. Unfortunately, we can not treat these processes at the macroscopic quantum level. On the other hand, we can avoid these details by simply introducing a suitable set of phenomenological boundary conditions. Thus, our theory will be fine at distances away from the interface larger than the size of a Cooper pair.

To see how all this works, consider a planar S/N interface as shown in Fig. 6.3. Assume the N material can be described by as a superconductor with $T >> T_c$. Then we have

$$\psi_n - \xi_n^2 \frac{d^2\psi_n}{dx^2} = 0 \tag{6.12}$$

where $\xi_n^2 = \hbar^2/2m^*\alpha_n$ and $\alpha_n > 0$ in the N region, and proximity effect.pdf

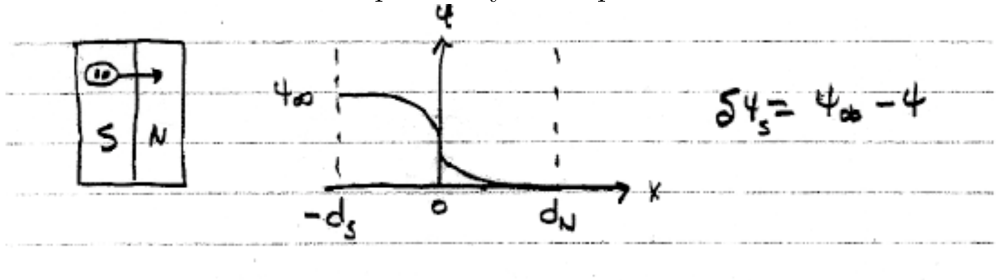


Figure 6.3: This is the caption

$$\delta\psi_s - \frac{\xi_s^2}{2}\frac{d^2\delta\psi_s}{dx^2} = 0 \tag{6.13}$$

in the S region.

The boundary conditions follow from the simple boundary condition found in the GL theory with an important addition:

$$\frac{1}{m_s^*}\,\psi_s\frac{d\psi_s}{dx} = \frac{1}{m_n^*}\,\psi_n\frac{d\psi_n}{dx} \tag{6.14}$$

and

$$\psi_n(0^+) = A\,\psi_s(0^-) \tag{6.15}$$

The first boundary condition is an obvious extension of that found in the GL theory. The second contains the new physics and permits a finite value of ψ to be induced in the N material.

Pictorally, these conditions can be visulized simply, and intuitively as seen in the figure below.

Note that in the GL approximation, ψ become discontinous at the boundary, reflecting our ignorance of the exact, microsopic (continuous) behavior.

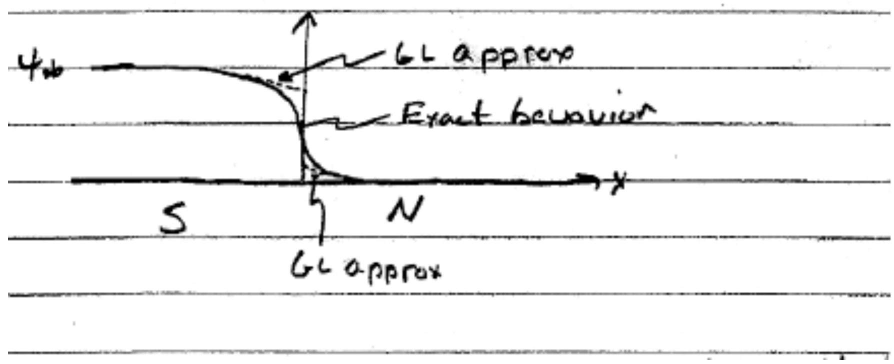
In any event, the problem now reduces to a simple set of linear first-order differential equations and their associated boundary conditions. From the form of the equations we can see that the solutions will be exponential functions, and from the boundary conditions, we have that the amplitudes of these exponential functions at the interface are given by effect BCs.pdf 

Figure 6.4: This is the caption

$$\frac{\delta\psi_s(0^-)}{\psi_\infty} = \frac{1}{1 + (\sqrt{2}/A^2)(m_n^*/m_s^*)(\xi_n/\xi_s)} \tag{6.16}$$

and

$$\frac{\psi_n(0^+)}{\psi_\infty} = \frac{1}{1 + (A^2/\sqrt{2})(m_s^*/m_n^*)(\xi_s/\xi_n)} \tag{6.17}$$

6.4 S/N domain wall energy

In our discussion of the intermediate state in Section 5.5, we introduced the notion of the domain wall energy between normal and superconducting domains within a superconductor. This domain wall energy can be calcuated straightforwardly using the GL theory, but the details are complicated. We shall use here a simpler qualitative argument and then simply quote the exact result.

Any domain wall energy is simply the excess energy necessary to form the wall. For an S/N doman wall in a superconductor, we know that the field will penetrate a distance λ from the normal region into the superconductor, and that the amplitude of the pair wave function must go from its equilibrium value ψ_∞ far from the boundary to zero in the normal region. The situation is illustrated in Fig. 6.5 below, along with the associated changes in the free energy density in the vicinity of the domain wall.

Just by looking a this figure, it is clear that domain wall schematic.pdf

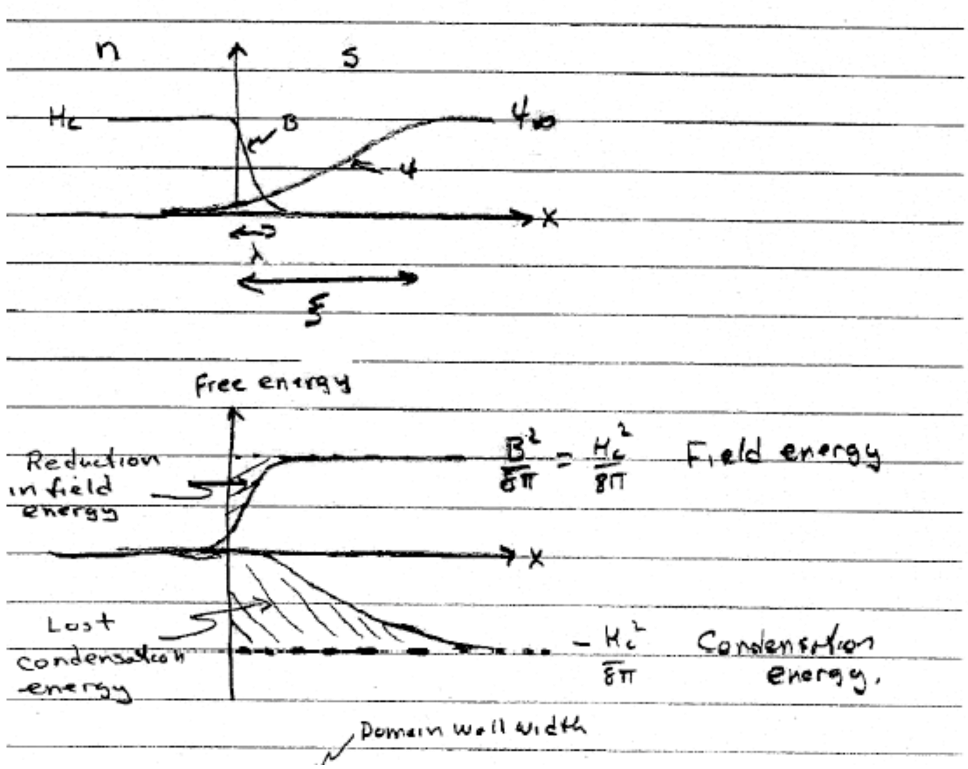


Figure 6.5: This is the caption

$$\gamma_{sn} = \frac{H_c^2}{8\pi}\,\delta_{sn} \approx \frac{H_c^2}{8\pi}\,(\xi - \lambda) \tag{6.18}$$

where $\delta_{sn} \approx (\xi - \lambda)$ is the width of the domain wall.

This is a striking result. It says that if $\xi < \lambda$, the domain wall energy is negative, which in turn means the the system will break up into N and S regions as finely as possible. Moreover, as we have already noted, this inequality holds for most superconductors, with the notable exception of the simple elemental superconductors. The exact dividing line is conventionally written in terms of the ratio

$$\kappa = \lambda/\xi \tag{6.19}$$

where the critical value at which the domain wall energy equals zero is given by $\kappa_c = 1/\sqrt{2}$. It follows therefore that for $\kappa < 1/\sqrt{2}$, one has a type 1 superconductor, and for $\kappa > 1/\sqrt{2}$, a type 2 superconductor.

6.5 Current-induced depairing

In general, as the current density J_s increases, the kinetic energy of a superconductor increases correspondingly. But this is not the whole story. As we shall show below, under these conditions, the pair density decreases so as to achieve the lowest free energy. For simplicity, consider a fine superconducting filament of diameter d and area S such that

$$d < \xi(T) \text{ and } \lambda(T) \tag{6.20}$$

The first conditon assures that ψ is constant across the filament, and the second that J_s is constant. Hence, we have a one-dimensioinal problem, and the free energy per unit length becomes

$$\frac{\mathcal{F}_s}{S} = \frac{\mathcal{F}_n}{S} + \alpha\,|\psi|^2 + \frac{\beta}{2}\,|\psi|^4 + \frac{\hbar^2}{2m^*}\left|\frac{d\psi}{dx}\right|^2 \tag{6.21}$$

Clearly, the energy will be minimized if $|\psi|$ does not vary along x. However, this does not mean that $|\psi| = \psi_0$ does not adjust in the face of an increasing current. In any event, it follows that

$$\psi = \psi_0\, e^{ikx} \tag{6.22}$$

and therefore

$$v_s = \frac{\hbar\nabla\phi}{m^*} = \frac{\hbar k}{m^*} \tag{6.23}$$

Fig. 6.6 shows this solution in both the macroscopic quantum (GL) picture and in the classical (London) picture.

Depairing cartoon.pdf

We can now write the free energy of the filament as

$$\frac{\mathcal{F}_s}{S} = \frac{\mathcal{F}_n}{S} + \alpha\,\psi_0^2 + \frac{\beta}{2}\,\psi_0^4 + \frac{1}{2}m^* v_s^2\,\psi_0^2 \tag{6.24}$$

which is natural, since we are using the Helmhotz free energy, for which v_s is the thermodynamic independent variable.

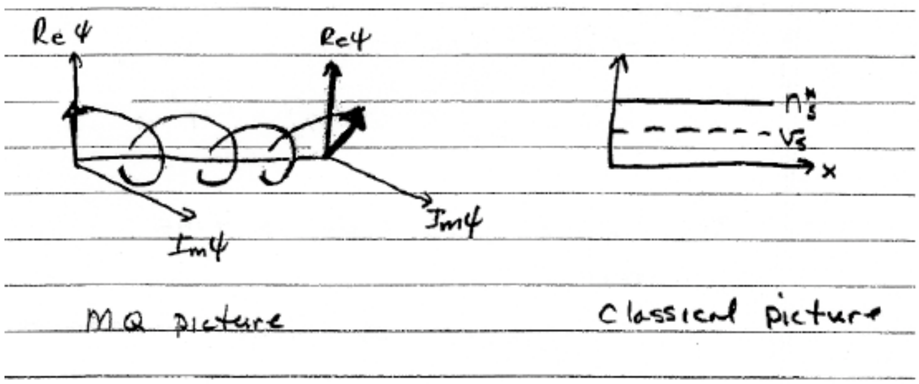
cartoon.pdf 

Figure 6.6: This is the caption

6.5.1 Behavior at fixed velocity

Taking v_s as the independent thermodynamic variable, we minimize the free energy above with respect to ψ_0^2, which leads to the condition

$$\alpha + \beta\psi_0^2 + \frac{1}{2}m^* v_s^2 = 0 \tag{6.25}$$

and therefore

$$\frac{\psi_0^2}{\psi_\infty^2} = \left(1 - \frac{1}{|\alpha|}\,\frac{1}{2}\,m^* v_s^2\right) \tag{6.26}$$

which can be written in two useful forms:

$$\frac{\psi_0^2}{\psi_\infty^2} = 1 - \left(\frac{v_s}{v_c}\right)^2 \tag{6.27}$$

where $v_c = (2|\alpha|/m^*)^{1/2} = \hbar/m^*\xi(T)$, and

$$\frac{\psi_0^2}{\psi_\infty^2} = 1 - (k\xi)^2 \tag{6.28}$$

where $k_c = \xi^{-1}$.

Note that as a function of v_s (or k), the pair density goes continuously to zero at a critical velocity v_c (or wave vector k_c). See Fig. 6.7 below.

The corresponding pair current is given by vs vsubs.pdf

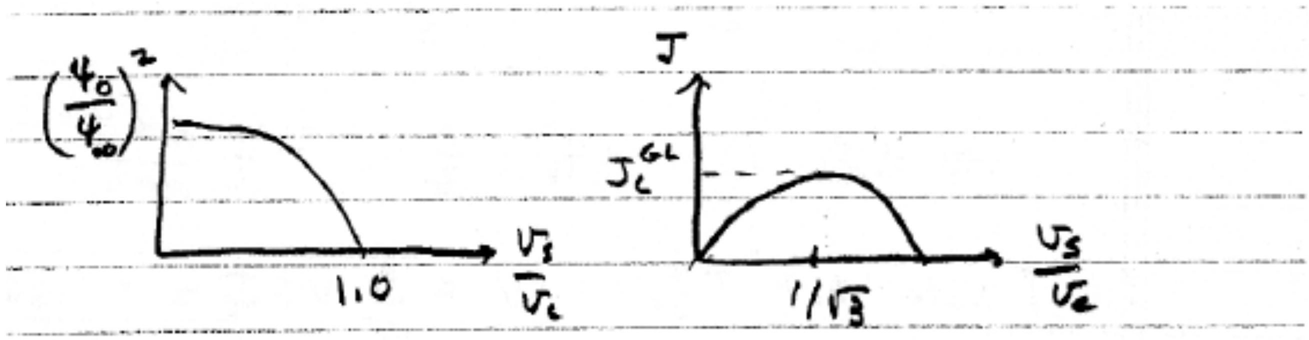


Figure 6.7: This is the caption

$$J_s = \psi_0^2\, e^*\, v_s = \psi_\infty^2 e^*\, v_c\Big(1 - \Big(\frac{v_s}{v_c}\Big)^2\Big)\Big(\frac{v_s}{v_c}\Big) \tag{6.29}$$

which has the striking property that J_s has a maximum

$$J_c^{GL} = \frac{2}{3\sqrt{3}}\,\frac{\psi_\infty^2\, e^*\hbar}{m^*\,\xi} \propto (1-t)^{3/2} \tag{6.30}$$

as a function of v_s. The current density J_c^{GL} is known as the GL depairing critical current density.

J_c^{GL} plays a very important role in superconductivity, as it represents the maximum possible current density a superconductor can carry, once all other factors limiting J_s have been circumvented, and the superconductor is only limited by the fundamental limit associated with its kinetic energy density. Another physical consequence of depairing is that since $n_s^* = n_s^*(J_s)$, it follows that the kinetic inductivity of a superconductor is nonlinear, $\mathcal{L}_\mathcal{K} = \mathcal{L}_\mathcal{K}(J_s)$.

Finally, we note that viewed as a function of $J_s < J_c^{GL}$, there must be two solutions for ψ, one on each side of the maximum in the function of J_s versus v_s. To see this explicitly and to better understand its meaning and significance, we need to examine the situation when J_s is the independent thermodynamic variable. This corresponds the the usual situation in practice in which the superconductor is current biased.

6.5.2 Behavior at fixed current density

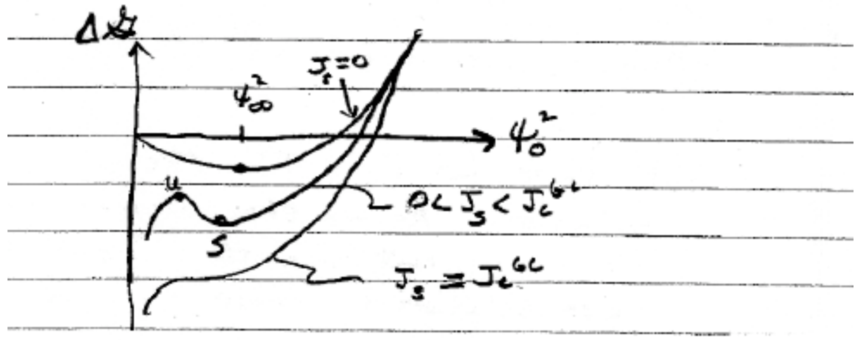
In the case that the current density is the independent thermodynamic variable, we must use the Gibbs free energy. For our filamentary superconductor, we have, using $\mathcal{G}_s = \mathcal{F}_s - (m^*/e^*)v_s J_s$,

$$\frac{\mathcal{G}_s}{S} = \frac{\mathcal{G}_n}{S} + \alpha\,\psi_0^2 + \frac{\beta}{2}\,\psi_0^4 + \frac{1}{2}m^* v_s^2\,\psi_0^2 - \frac{m^*}{e^*}v_s J_s \tag{6.31}$$

or

$$\frac{\mathcal{G}_s}{S} = \frac{\mathcal{G}_n}{S} + \alpha\,\psi_0^2 + \frac{\beta}{2}\,\psi_0^4 - \frac{1}{2}\frac{m^* J_s^2}{e^*\psi_o^2} \tag{6.32}$$

Fig. 6.5.2 shows the functional dependence of $\mathcal{G}_s/S$, from which it is readily seen that, for any $0 < J_s < J_c^{GL}$, there are two solutions (extrema) for ψ_0, one stable and the other unstable. The unstable solution corresponds to the high velocity side of the J_s vs v_s curve in Fig. 6.7 above.



for depairing.pdf

Figure 6.8: This is the caption

But the significance of Fig. 6.5.2 is far greater than simply confirming that, for a given J_s, there are two solutions for ψ_0. It shows that the pair current in a superconductor is only metastable to a collapse of ψ_0 to zero. Of course, strictly speaking, for our filament in the thermodynamic limit (infinitely long filament) the energy barrier for a uniform collapse, which is proportional to the length of the filament, is infinite.

But what if we were to relax the assumption that $|\psi|$ is constant along the length of the filament, and allow local fluctuations? As we shall analyze later in detail when we deal with fluctuations in superconductors, in this case the energy barrier for a filament becomes finite and locally $|\psi|$ can fluctuate to zero. But if $|\psi| \rightarrow 0$, J_s also goes to zero, and superconductivity will reform. Rougly speaking, the superconductor would find itself at the origin in Fig. 6.5.2, and ψ_0 would naturally increase toward ψ_∞ The actual picture is more subtle than this, but the notion of a dynamic cyclic state is basically correct and is part of the story about how we understand resistance in superconductors. Needless to say, we will return to such important issues in great detail. For now, let us simply complete the details of the calculation at hand.

To find $\psi_0(J_s)$, we once again minimize our free energy with respect to ψ_0^2, which yields the condition,

$$\alpha\psi_0^4 + \beta\psi_0^6 + \frac{1}{2}\frac{m^* J_x^2}{e^*} = 0 \tag{6.33}$$

the solutions of which can be visualized in Fig. 6.5.2 below.

solution .pdf

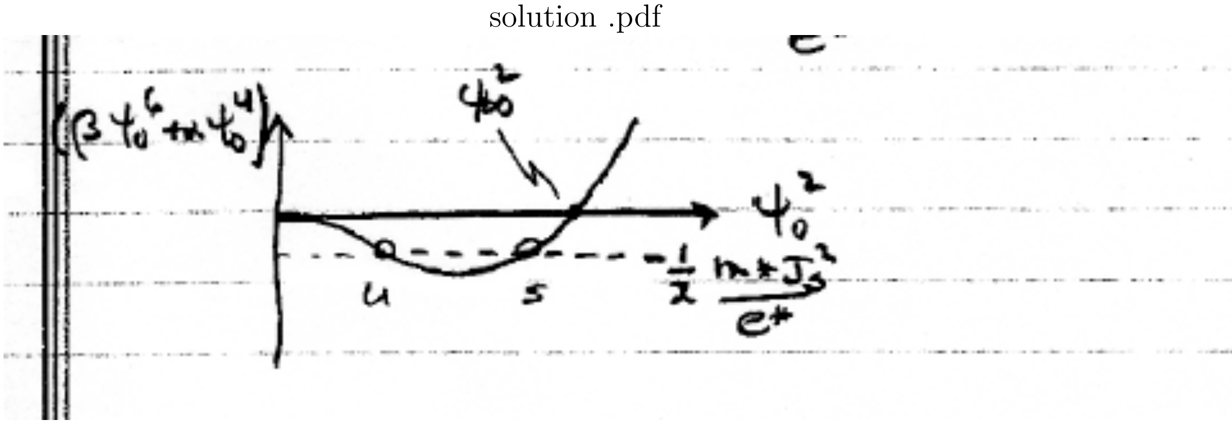


Figure 6.9: This is the caption

To obtain the maximum of J_s, we determine the point of instability given by

$$\frac{\partial^2 \Delta\mathcal{G}_s}{\partial(\psi_0)^2} = \beta - \frac{m^* J_c^{GL\,2}}{e^*\psi_0^6} \tag{6.34}$$

where $\Delta\mathcal{G}_s = \mathcal{G}_s - \mathcal{G}_n$, and the instability occurs at

$$\frac{\psi_0^2(J_c^{GL})}{\psi_\infty} = \frac{2}{3} \tag{6.35}$$

and

$$J_c^{GL} = \frac{2}{3\sqrt{3}} \frac{\psi_\infty^2 \, e^* \hbar}{m^* \, \xi} \tag{6.36}$$

as before.

6.5.3 Depairing in a multiply connected superconductor

Depairing in multiply connected superconductors has some special twists. First, when a superconductor is multiply connected, under certain conditions, it permits independent control of the velocity of the pairs. Second, it leads to a very interesting phase diagram for the superconductor in an applied magnetic field – the so-called Little-Parks effect.

Velocity control

Consider a thin ($d < \lambda$), hollow cylinder in a parallel applied field.

in hollow cylinder.pdf

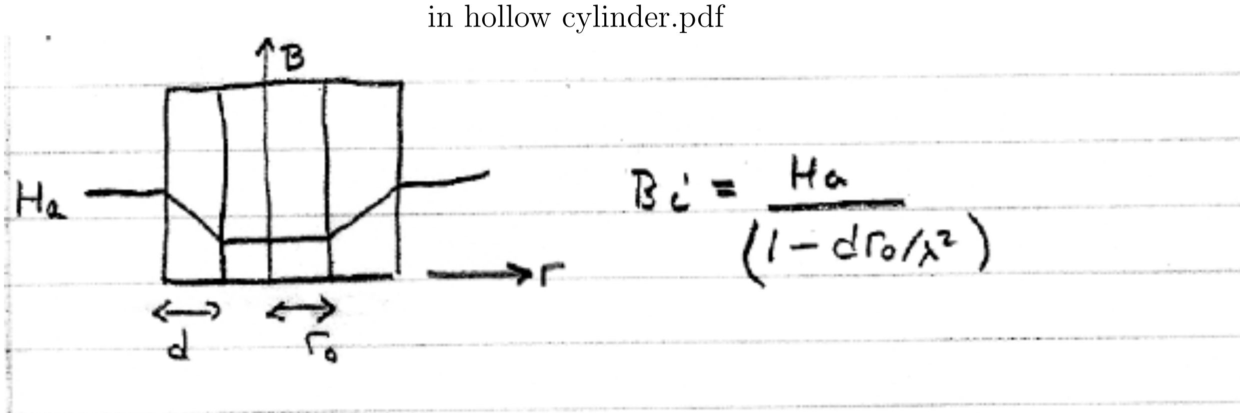


Figure 6.10: This is the caption

As we have shown previously, the field in the interior of the cylinder is given by

$$h_i = \frac{H_a}{(1 - dr_0/\lambda^2)} \tag{6.37}$$

From fluxoid quantization, we have

$$\oint (m^* v_s + \frac{e^*}{c} A) \cdot dl = nh \tag{6.38}$$

which implies that

$$\frac{2\pi r_0 c}{e^*} m^* v_s + \Phi_i = n\Phi_0 \tag{6.39}$$

or

$$v_s = \frac{\hbar}{m^* r_0}\Big(n - \frac{\Phi_i}{\Phi_0}\Big) \tag{6.40}$$

From this result, we see that the velocity v_s can be controlled by H_a to the degree that $\Phi_i \approx \Phi_a = \pi {r_0}^2 H_a$, which requires $dr_0/\lambda^2 << 1$. Recall, however, that as $v_s \to v_c$, $n_s^* \to 0$, and therefore $\lambda \to \infty$, which means that the required inequality is satisfied automatically when the depairing is strong. So, we see that in practice it is possible to control v_s.

Returning now to our free energy density

$$\Delta\mathcal{F}_s = \alpha\, \psi_0^2 + \frac{\beta}{2}\, \psi_0^4 + \frac{1}{2} m^* v_s^2\, \psi_0^2 \tag{6.41}$$

it is convenient to transform to reduced variables, $f = \psi_0/\psi_\infty$ and $v = v_s/v_c$, in which case the free energy density becomes

$$\Delta\mathcal{F}_s = \frac{H_c^2}{4\pi}(- f^2 + \frac{1}{2} f^4 + v^2 f^2) \tag{6.42}$$

Now substituting $f^2 = (1 - v^2)$ corresponding to the free energy minima, we arrive at

$$\Delta\mathcal{F}_s(v) = -\frac{H_c^2}{8\pi}(1 - 2v^2 + v^4) \tag{6.43}$$

which is depicted in Fig. 6.5.3 below, where we see that as $v \to 1$, the condensation energy of the superconductor is continuously depleted. Note, however that the curve shown is not a simple parabola, as it would be in the London limit. The exact shape reflects the gradual depairing.

energy vs v.pdf

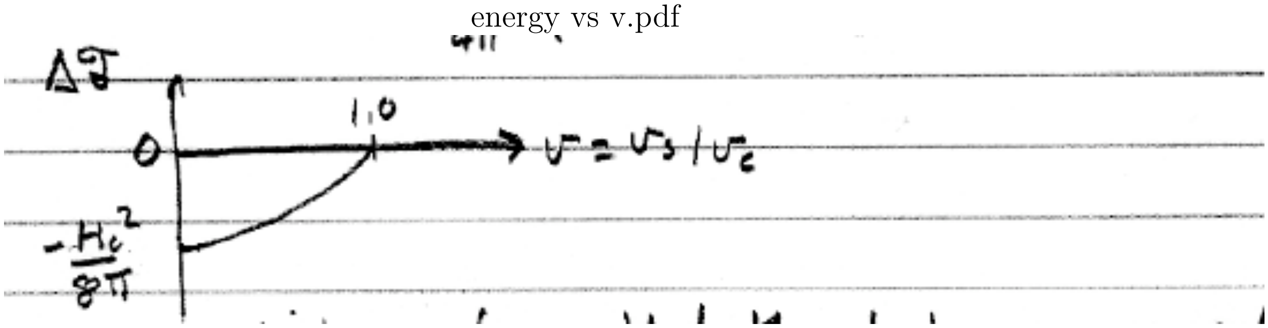


Figure 6.11: This is the caption

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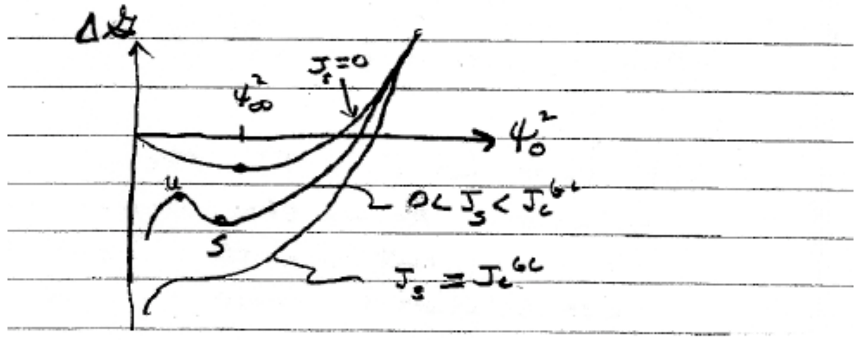

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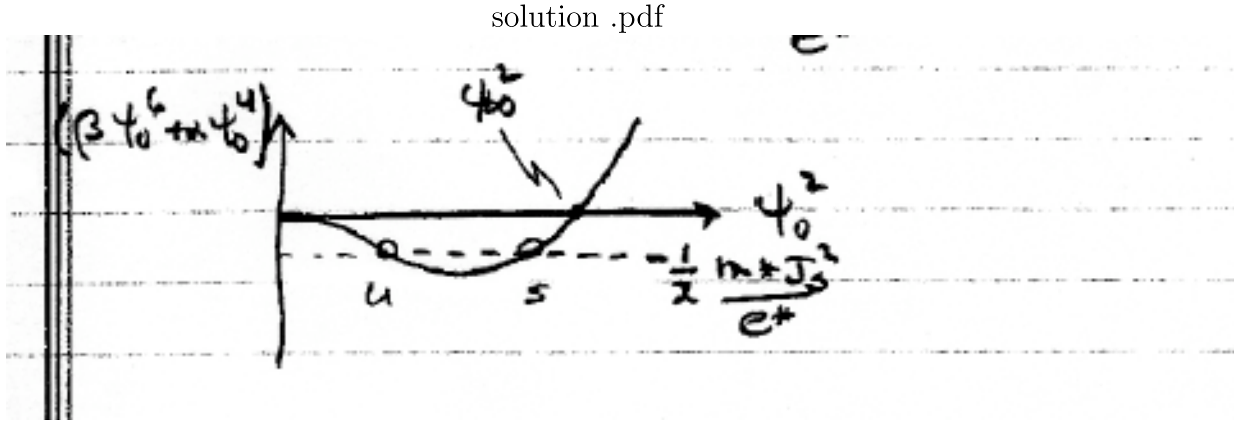

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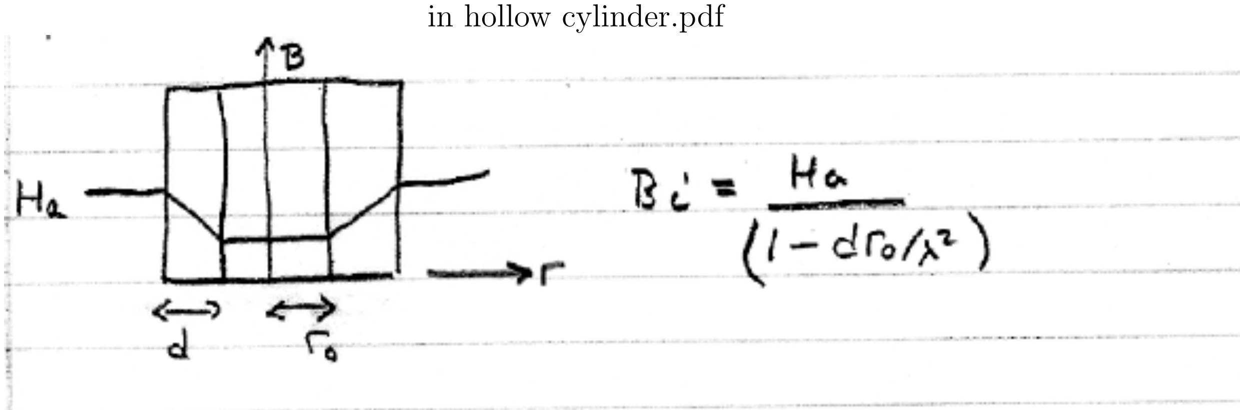
6.6 Multiply connected superconductors

Depairing in multiply connected superconductors has some special twists. First, when a superconductor is multiply connected, under certain conditions, it permits independent control of the velocity of the pairs. Second, it leads to a very interesting phase diagram for the superconductor in an applied magnetic field – the so-called Little-Parks effect.

6.6.1 Velocity control

Consider a thin ($d < \lambda$), hollow cylinder in a parallel applied field.

in hollow cylinder.pdf



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energy vs v.pdf

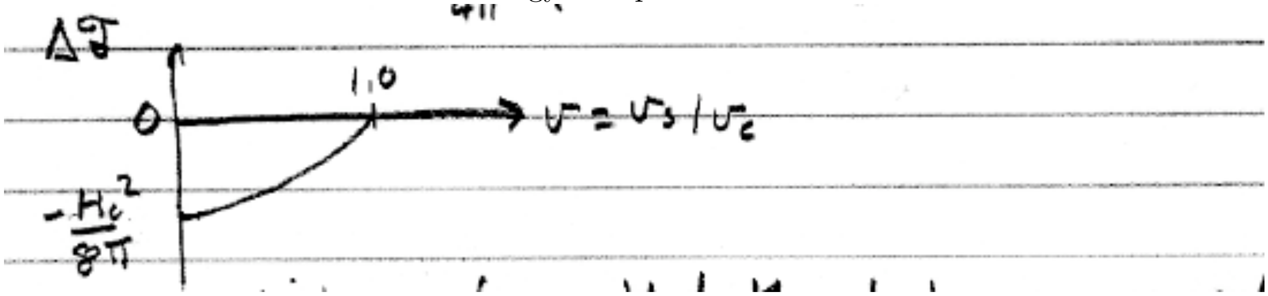


Figure 6.11: This is the caption

6.6.2 Phase diagram – The Little-Parks effect

Depairing in a multiply-connected superconductor has a very interesting physical effect on the phase diagram of the superconductor. Consider the cylinder in the normal state and ask at what temperature will the cylinder go superconducting as the temperature is reduced. Since

$$v_s = \frac{\hbar}{m^* r_0}\Big(n - \frac{\Phi_i}{\Phi_0}\Big) \tag{6.44}$$

it follows that for $\Phi_i \neq n\Phi_0$, the superconducting state must form at finite pair velocity. Thus, T_c must be reduced to permit enough condensation energy to compensate for the needed kinetic energy. At the transition $\lambda \to \infty$, and therefore $\Phi_i = \Phi_a = H_a \pi {r_0}^2$, and the magnetic field energy term in the free energy is zero.

Under these conditions,

$$\Delta\mathcal{F}_s(\Phi_a) = \alpha\,\psi_0^2 + \frac{\beta}{2}\,\psi_0^4 + \frac{1}{2}\psi_0^2 m^*\Big(\frac{\hbar}{m^* r_0}(n - \frac{\Phi_a}{\Phi_0})\Big)^2 \tag{6.45}$$

or

$$\Delta\mathcal{F}_s(\Phi_a) = \Big(\alpha - \frac{\hbar^2}{2m^* r_0^2}\Big(n - \frac{\Phi_a}{\Phi_0}\Big)^2\Big)\psi_0^2 + \frac{\beta}{2}\,\psi_0^4 \tag{6.46}$$

from which it is obvious that the reduced $T_c(\Phi_a)$ is governed by the relation

$$\alpha_0\Big(\frac{T_c(\Phi_a) - T_{c0}}{T_{c0}}\Big) - \frac{\hbar^2}{2m^* r_0^2}\Big(n - \frac{\Phi_a}{\Phi_0}\Big)^2 = 0 \tag{6.47}$$

defined by the condition that the coefficient of the ψ_0^2 term in the free energy goes through zero. Here T_{c0} is the transition temperature in zero applied field.

Rearranging terms, we arrive at

$$\frac{T_c(\Phi_a/\Phi_0)}{T_{c0}} = 1 - \left(\frac{\xi(0)}{r_0}\right)^2 \left(n - \frac{\Phi_a}{\Phi_0}\right)^2 \tag{6.48}$$

which leads to the phase diagram shown below in Fig. 6.12 . This oscillation in the T_c of the cylinder is known as the Little-Parks effect. Note that if $\xi(0) > 2r_0$, $T_c(\Phi_a)$ can go all the way to zero, leading to the reenterant phase diagram also shown in the figure. This is a striking result. It says that, under these conditions, in an increasing applied field at zero temperature, the cylinder alternately goes normal and then superconducting again, each time forming in a higher quantum state indexed by n. Because $\xi(0)$ can exceed a micron or so in some low temperature superconductors, this effect is in fact observable.

Another interesting perspective on the behavior of a superconducting cylinder is obtained by considering the free energy density as a function of Φ_a/Φ_0, as shown in Fig. 6.13 below.

As is evident, for $\xi(0) < 2r_0$, the cylinder has multiple metastable states corresponding to different quantum numbers n. So, under this condition, if we increase Φ_a, the system may or may not make a transition to the next quantum state at $\Phi_a = \Phi_0/2$, depending on how such a transition may be nucleated. Such metastable states can be quite stable, as evidenced by the stability of persistent currents in rings discussed in Section 1.1. However, when $\xi(0) > 2r_0$, there is no question that the transitions $(n \rightarrow n+1)$ occur regularly as Φ_a increases. In any event, as this figure makes clear, the decay of persistent currents is associated with transitions between the quantum states of the system. We return later in these notes to the mechanism of these transitions.

6.7 Strong field effects in the Meissner state

In the Meissner state, where there is no flux penetration, we have superconductivity in its purest form. And, as we have seen, for weak fields and current densities, the Meissner state is well described by the London theory. On the other hand, as the applied field increases, so to does the current density near the surface, and depairing arises. To see this, recall that the critical current density at the surface of a type 1 superconductor in a field $H_a = H_c$ is given by

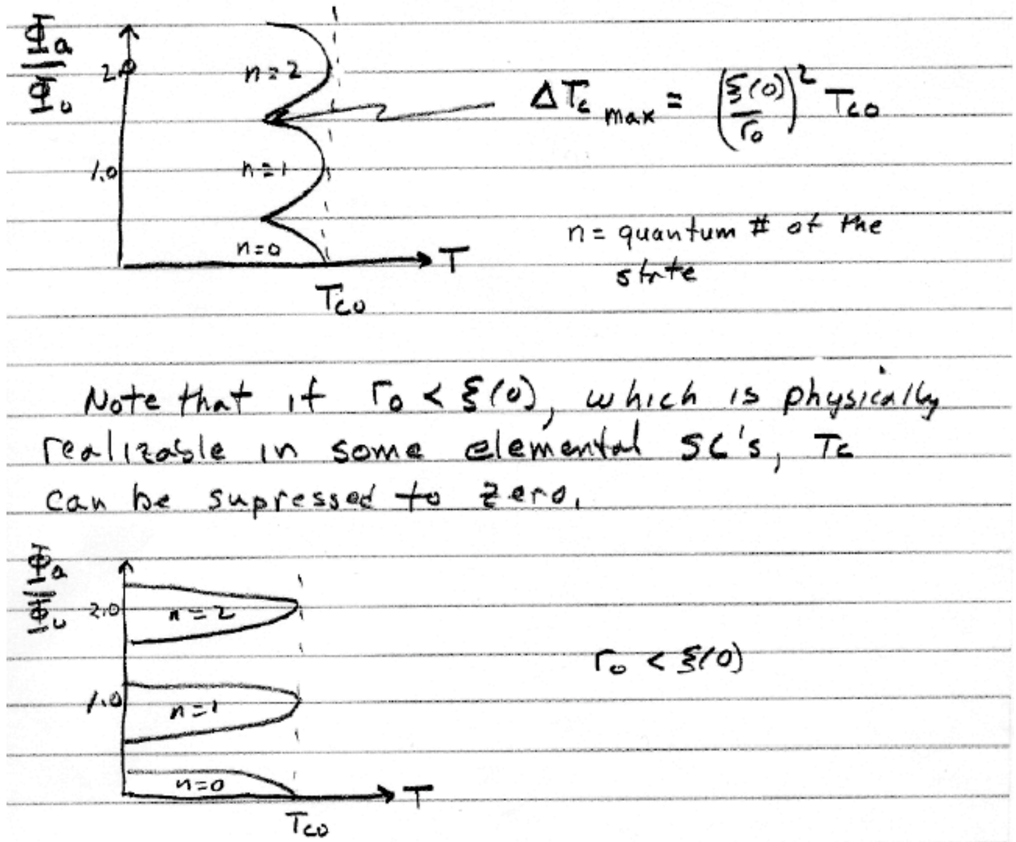


Figure 6.12: This is the caption

$$J_s = \frac{c}{4\pi}\frac{H_c}{\lambda} \tag{6.49}$$

which is comparable to the GL depairing critical current density

$$J_c^{GL} = \frac{2}{3\sqrt{3}}\frac{\psi_\infty^2\, e^*\hbar}{m^*\,\xi} = \frac{4}{3\sqrt{6}}\frac{c}{4\pi}\frac{H_c}{\lambda} \tag{6.50}$$

Therefore, a complete description of the Meissner state requires a theory that goes beyond the simple London theory. However, as we shall now show, it is not necessary to go to the full GL theory. It is possible to modify slightly the London theory to account for these strong field effects.

Let us begin with the full GL free energy density written in reduced units energy vs phi-suba.pdf

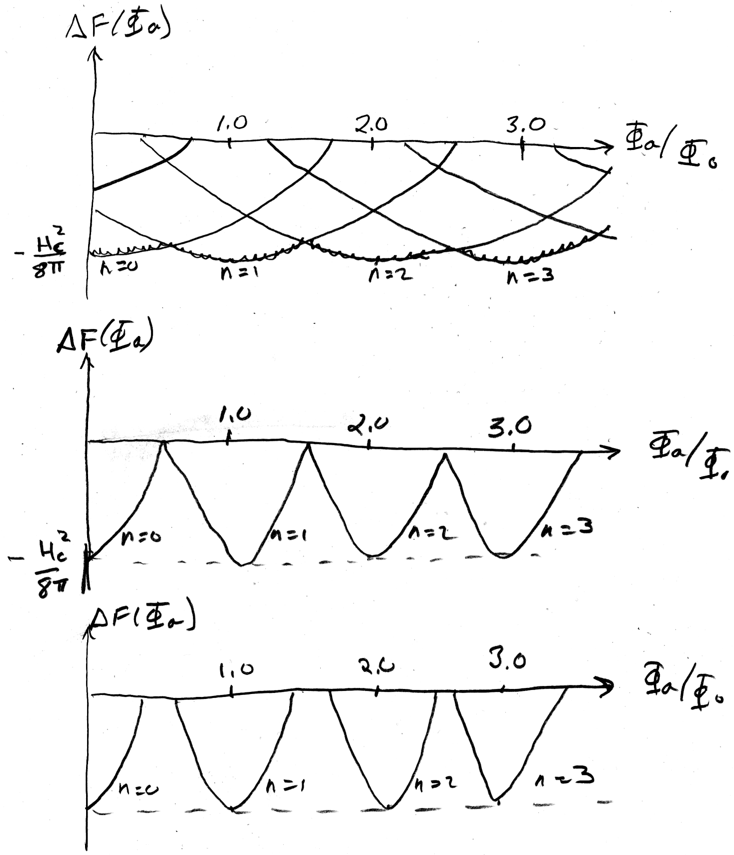


Figure 6.13: This is the caption

$$\Delta\mathcal{F}_s = \frac{\alpha^2}{\beta}\Big(- |f|^2 + \frac{1}{2}|f|^4 + \xi^2|(-i\nabla - \frac{e^*}{\hbar c}A)f|^2\Big) \tag{6.51}$$

Examining the kinetic energy term, we see that

$$\xi^2\Big|(-i\nabla - \frac{e^{*2}}{\hbar c}A)f\Big|^2 = \xi^2\Big((\nabla|f|)^2 + f^2(\nabla\phi - \frac{e^*}{\hbar c}A)^2\Big) \tag{6.52}$$

Now as $v_s \to v_c$, depairing arises near the surface over a length scale λ. Therefore, the energies involved in the kinetic energy are of order

$$\xi^2\Big(\frac{1}{\lambda}+\frac{m^{*2}v_c^2}{\hbar^2}\Big)=\frac{\xi^2}{\lambda^2}+1 \tag{6.53}$$

which shows that we can neglect the gradient term and keep only the classical kinetic energy term, particularly since λ diverges as $v_s \to v_c$.

$$\Delta\mathcal{F}_s=\frac{\alpha^2}{\beta}\Big(-f^2+\frac{1}{2}f^4+f^2(\nabla\phi-\frac{e^*}{\hbar c}A)^2\Big) \tag{6.54}$$

Clearly, the following situation now prevails. We have essentially the London limit but where the density of pairs undergoes depairing according to the equation $f^2=(1-v^2)$, and where v is the reduced velocity. The net result is a non-linear London theory in which $\lambda=\lambda(f^2)$ is determined self-consistently with the pair velocity v. Obviously, under these conditions the decay of the field and current density in the superconductor is no longer exponential.

6.8 The Josephson effect and weakly coupled superconductors

The Josephson effect is one of most profound consequences of the macroscopic quantum nature of superconductivity. And, it is actually very simple. Consider two superconductors that are brought into weak electrical contact, such that Cooper pairs can pass back and forth between the two superconductors. Before establishing contact, the two superconductors are independent and have independent quantum phases (for simplicity we ignore fields and currents internal to the two superconductors). Once contact is made, they must become one superconductor with one phase. Viewed this way, the Josphson effect is simply that process by which two superconductors couple their quantum phases in the presence of a weak coupling. The situation is illustrated in Fig. 6.14 below.

6.8.1 The Josephson equations

To model this process requires only a simple extension of the GL theory. For sufficiently weak coupling, we can expand the coupling energy in a Taylor series expansion in terms of the difference between the pair wave functions of the two superconductors. Keeping only the leading term, which defines weak coupling, we have

$$F_J=\eta|\psi_1-\psi_2|^2 \tag{6.55}$$

of Josephson coupling.pdf

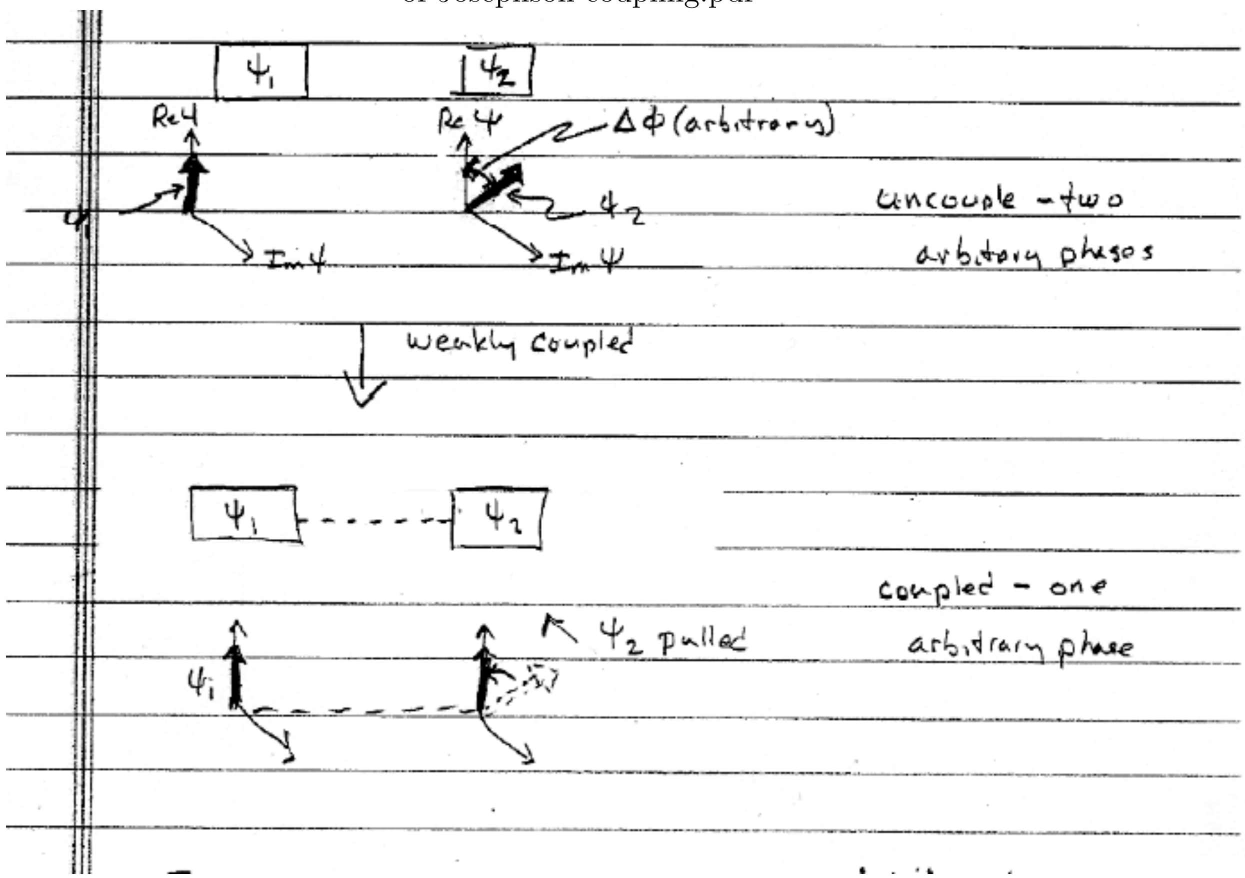


Figure 6.14: This is the caption

where for the moment, we assume the coupling occurs at a point. Here η is a phenomenological coefficient that gauges the strength of the coupling. Clearly, this difference term is nothing more than the discrete analog of the gradient term in the GL free energy.

Assuming identical superconductors ($|\psi_1| = |\psi_2|$), and writing the ψ's in terms of their modula and phases, the Josephson coupling energy can be written

$$F_J = \eta|\psi|^2(1 - cos\Delta\phi) = F_{J0}(1 - cos\Delta\phi) \tag{6.56}$$

where $\Delta\phi$ is the difference in the phases across the junction.

Thus we are led to the result that the coupling energy is a periodic function of the difference is phases across the junction. This is the Josephson effect. Moreover, we emphasize that this is a completely general result with no reference to the specific nature of the weak coupling. It can be due to tunneling, the proximity effect, a point contact or any other means of weak coupling.

To determine the current through the junction is a straightforward application of the thermodynamic relation

$$I = \frac{e^*}{\hbar}\frac{dF}{d(\Delta\phi)} \tag{6.57}$$

derived in Section 4.5

The result is

$$I = \frac{e^*}{\hbar}F_{J0}\sin\Delta\phi = I_{J0}\sin\Delta\phi \tag{6.58}$$

where $I_{J0} = (e^*/\hbar)F_{J0}$ and has the physical interpretation as the critical current of the junction. The periodic nature of $F_J(\Delta\phi)$ and $I_J(\Delta\phi)$ are illustrated in Fig. 6.15, which also shows the configuration of the two pair wave functions on the two sides of the junction as the phase difference evolves.

dependence of F and J.pdf

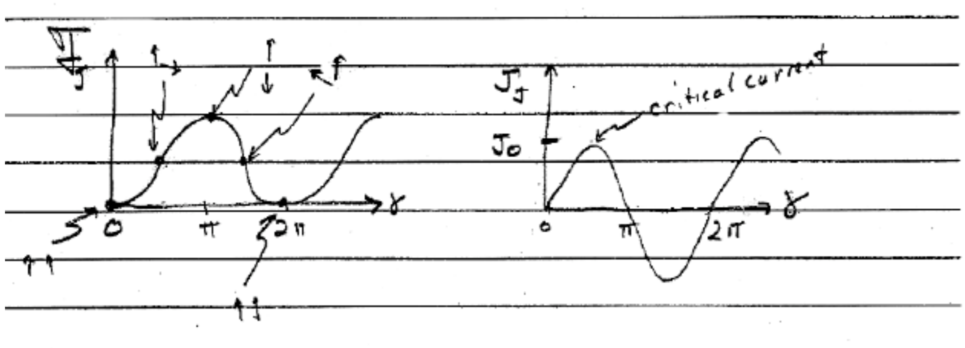


Figure 6.15: This is the caption

To include the effect of magnetic fields, recall that it is only necessary to replace $\Delta\phi$ by the gauge invariant phase difference

$$\gamma = \Delta\phi - \frac{e^*}{\hbar c}\int_1^2 \vec{A}\cdot\vec{dl} \tag{6.59}$$

and where the time rate of change of γ is related to the voltage across the junction by

$$\frac{d\gamma}{dt} = \frac{e^*}{\hbar c} V \tag{6.60}$$

The physical properties of Josephson junctions that follow from the Josephson equations derived here are truly extraordinary. We will return to these properties in detail in Chapter xxxx.

In closing this discussion, we note that the results above generalize immediately to junctions that are not point junctions. One simply replace the current $I_J(\gamma)$ by the current density $J_J(\gamma(x,y))$ where x and y are the coordinates in the plane of the junction. In this case the Josephson coupling energy can be though of as an interface energy between the two superconductors.

6.8.2 Some specific types of junctions

The derivation of the Josephson effect above is completely general. It is helpful in deepening understanding to consider in addition some explicit examples. Here we consider coupling via the proximity effect, and through a short filamentary superconductor.

Proximity effect Josephson junctions

We have already discussed the proxmity effect. Our task here is simply to apply that knowledge to and SNS Josephson junction. Consider a planar junction as shown schematically in cross section in Fig. 6.16 . Assuming a proximity effect model for the N region, we have immediately that in N, ψ_n is governed by the differential equation

$$\xi_n^2 \frac{d^2\psi_n}{dx} = \psi_n \tag{6.61}$$

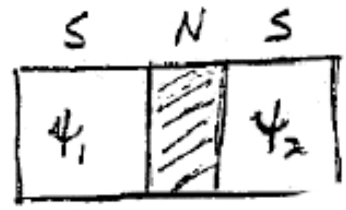
where ξ_n is the normal state coherence length.

Clearly, in N the pair wave function can be written in general

$$\psi_n(x) = Ae^{-x/\xi_n} + Be^{x/\xi_n} \tag{6.62}$$

where A and B are complex coefficients to be determined from the boundary conditions. We take the boundary conditions just on the N side of the two SN interfaces

$$\psi_n(0^+) = \psi_{n1} \quad \text{and} \quad \psi_n(l^-)e^{i\Delta\phi} \tag{6.63}$$



Josephson junction schematic.pdf

Figure 6.16: This is the caption

where $\Delta\phi$ is the phase difference across the junction.

Using these boundary conditions, we get

$$\psi_n(x) = |\psi_n(x)|e^{i\phi(x)} = \frac{|\psi_{n1}|\sinh(l-x)/\xi_n + |\psi_{n2}|\sinh(x/\xi_n)e^{i\Delta\phi}}{\sinh(l/\xi_n)} \tag{6.64}$$

which, in the limit $l >> \xi_n$, has the intuitive form

$$\psi_n(x) = |\psi_{n1}|\, e^{-(l-x)/\xi_n} + |\psi_{n2}|\, e^{(-x/\xi_n)}e^{i\Delta\phi} \tag{6.65}$$

i.e., complex decaying exponentials from both SN interfaces.

wave function at O and Pi.pdf

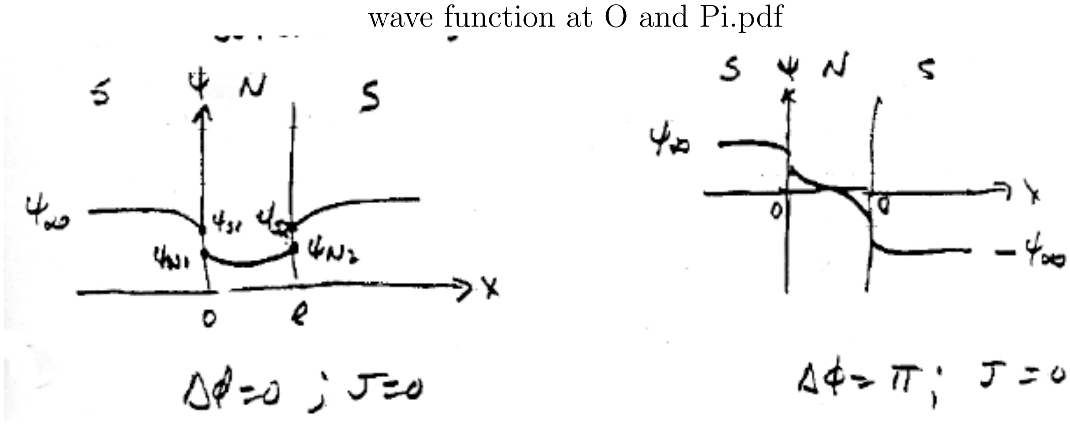


Figure 6.17: This is the caption

The behavior of $\psi_n(x)$ for $\Delta\phi = 0, \pi$ is shown in the Fig. 6.17 below. Note that in both cases $J_s = 0$, and that for $\Delta\phi = \pi$, $\psi_n(x)$ goes through zero inside the junction. Indeed, in general, ψ must go through zero for $\Delta\phi = \pi$, no matter how the coupling is achieved. This follows from the fact that if $J_s = 0$, there can be no spatial gradients of ϕ and therefore ψ can be taken as real. But if this real ψ must go from positive to negative as it crosses the junction, it follows necessarily that ψ must be zero somewhere inside the junction. And, indeed, this characteristic is central to the physics of a Josephson junction. How else could the properties be a periodic function of $\Delta\phi$? For a more elaborated discussion of this point, see Section xxx below on the distinction between phase winding and phase slippage.

Given the pair wave function, it is straight forward to calculate the current density in the junction

$$J_s(x) = \frac{e^*\hbar}{m^*} Im(\psi^*(x)\nabla\psi(x)) \tag{6.66}$$

$$= \frac{e^*\hbar}{m^*} \frac{|\psi_{n1}||\psi_{n2}|}{\xi_n} \frac{1}{\sinh(l/\xi_n)} \sin\Delta\phi = J_{J0} \sin\Delta\phi \tag{6.67}$$

which is independent of x as required by current conservation.

In the limit $l >> \xi_n$, this reduces to

$$= \frac{e^*\hbar}{m^*} \frac{|\psi_{n1}||\psi_{n2}|}{\xi_n} e^{-l/\xi_n} \sin\Delta\phi \tag{6.68}$$

which, as expected, is periodic in $\Delta\phi$, and which shows explicitly that the junction critical current density falls off exponentially with the thickness of the junction. Also, note that although Eqn xxx above gives an explicit expression for J_{J0}, it is written in terms of ψ on the N sides of the two SN boundaries. To know these values in terms of ψ_∞ in the superconductors requires knowledge of the value of A in the proximity effect boundary conditions introduced in Section 6.3, which, in general, we do not know. In practice, it is best to take J_{J0} as an experimental parameter.

Short superconducting bridge with rigid boundary conditions

As our second example, we take a short superconducting bridge (filament of length $l < \xi(T)$) between two massive superconducting banks. This entire structure is one superconductor, but still it acts as a weak link, as we shall see. To model this case, we use the GL theory to calculate ψ along the bridge using so-called rigid boundary conditions. The term rigid refers to the fact that when a one dimensional superconducting filament is in contact with large (three dimensional) bank, the pair wave function in the bank is unperturbed by the behavior in the bridge. Mathematically, this means that there is no healing of ψ in the banks, and we can take $\psi = \psi_\infty$ (with a possible phase difference) as the boundary condition at the two ends of the bridge.

Consider now the free energy of the filament in reduced units,

$$F_J = S \int_{-l/2}^{+l/2} dx\, \frac{H_c^2}{4\pi} \Big(-|f|^2 + \frac{1}{2}|f|^4 + \xi(T)^2 |\nabla f|^2 \Big) \tag{6.69}$$

For a phase difference of $\Delta\phi$ across the bridge, the (symmetrized) boundary conditions become

$$f(-l/2) = e^{-i\Delta\phi/2} \quad \text{and} \quad f(l/2) = e^{+i\Delta\phi/2} \tag{6.70}$$

Clearly, as f varies across the bridge in response to the phase difference, the kinetic energy dominates the free energy density

$$\xi^2 |\nabla f|^2 \quad \rightarrow \quad \mathcal{O}(\xi/l)^2 >> 1 \tag{6.71}$$

Under these conditions, the first GL equation reduces to

$$\nabla^2 f = 0 \tag{6.72}$$

which has the solution

$$f = A + Bx \tag{6.73}$$

where again A and B are complex coefficients to be determined by the boundary conditions.

Straight forward manipulation leads to

$$f(x) = \frac{-2x + l}{2l} e^{-i\Delta\phi/2} + \frac{2x + l}{2l} e^{i\Delta\phi/2} \tag{6.74}$$

which is illustrated in Fig. 6.18 below and can be visualized as a straight line between the pair wave functions on either end of the bridge. Note also that just as for the proximity junction case, $\psi(x)$ goes through zero for $\Delta\phi = \pi$, as we argued before it must do in general.

And, once again, given $\psi(x) = \psi_\infty f$, J_s can be directly calculated

$$J_s(x) = \frac{e^* \hbar}{m^*} Im(\psi^*(x) \nabla \psi(x)) \tag{6.75}$$

$$= J_{J0} \sin \Delta\phi \tag{6.76}$$

fix equation

bridge wave function.pdf

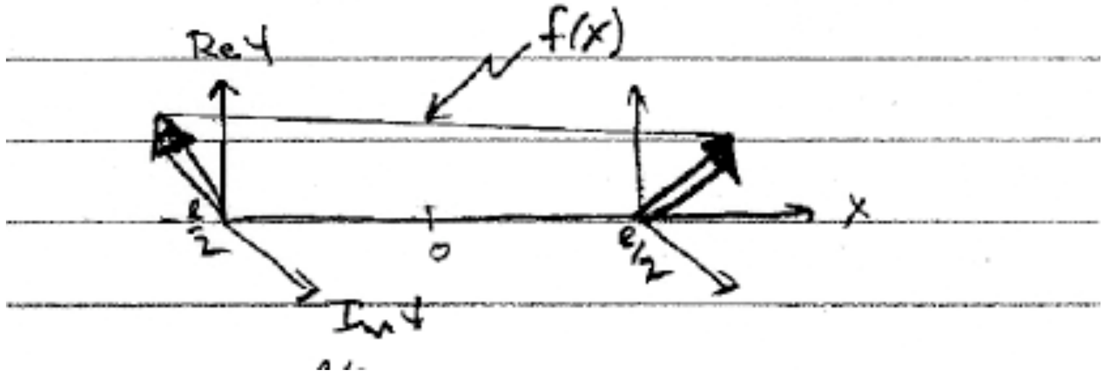


Figure 6.18: This is the caption

6.8.3 Josephson coupling between non-s-wave superconductors

As we have emphasized, the Josephson effect is fundamentally a phenomenon associated with the interface between two superconductors. Therefore, the bulk symmetry arguments that make the GL theory the same independent of the orbital state of the pairs themselves do not apply. Indeed, as we shall show, the symmetry of the internal structure of the pairs enters fundamentally into the Josephson coupling.

To see this, let us go back to our expression for the pair wave function

$$\psi_{sc} = Y_{lm}(\theta, \phi)\psi(r) \tag{6.77}$$

In terms of ψ_{sc}, the Josephson coupling energy becomes

$$F_J(\Delta\phi) = \int d\cos\theta_1 d\phi_1 d\cos\theta_2 d\phi_2 F'_J(\psi_{sc1} - \psi_{sc2}) \tag{6.78}$$

where

$$F'_J(\Delta\phi;\theta_1,\phi,\theta_2\phi_2) = \eta'(\theta_1,\phi_1;\theta_2,\phi_2)|Y_{l_1m_1}\psi_1 - Y_{l_2m_2}\psi_2|^2 \tag{6.79}$$

The essential part of the Josephson coupling involves the cross terms between ψ_{sc1} and ψ_{sc2}, and it is just these terms that reflect the internal nature of the pairs. Specifically, the phase difference dependent part of $F'_J(\Delta\phi)$

$$= \eta'(\theta_1,\phi_1;\theta_2,\phi_2)\Big(Y^*_{l_1m_1}Y_{l_2m_2}\psi^*_1\psi_2 - Y_{l_1m_1}Y^*_{l_2m_2}\psi_1\psi^*_2\Big) \tag{6.80}$$

which contains the information about how the internal structure of the pair wave function effects the coupling from one superconductor to the other. After integrating over the internal coordinates of the pairs, this becomes

$$= Re(\eta\, \psi_1^* \psi_2) \tag{6.81}$$

where

$$\eta = \int d\cos\theta_1 d\phi_1 d\cos\theta_2 d\phi_2 \left(Y^*_{l_1 m_1}(\theta_1, \phi_1)\, \eta'(\theta_1, \phi_1; \theta_2, \phi_2)\, Y_{l_2 m_2}(\theta_2, \phi_2) \right) \tag{6.82}$$

which contains the information about how the internal wave functions of the pairs "overlap" across the junction. The details here are obviously complicated and depend on the microscopic processes at the interface. But, some intuitive insight can be obtained from the cartoons shown below, which illustrate the situation for coupling between two s-wave superconductor and an s-wave and a d-wave superconductor. For simplicity, the two cartoons illustrate coupling between two two-dimensional superconductors. By simple symmetry considerations, one can see that in some cases there will be Josephson coupling and in others not.

Josephson coupling.pdf

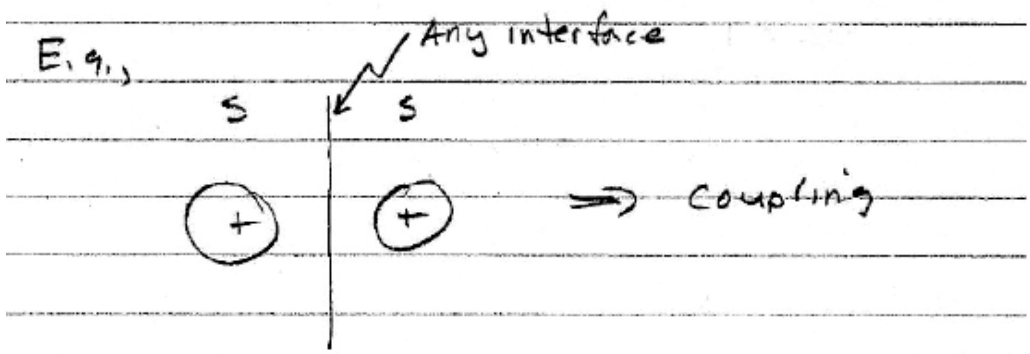
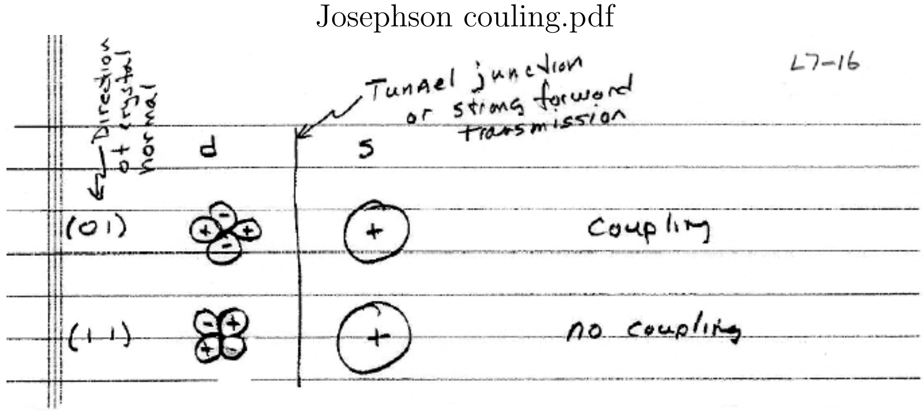


Figure 6.19: This is the caption

Josephson couling.pdf



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